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AN

ELEMENTARY TREATISE

ON

OPTICS:

PART I.

CONTAINING

ALL THE REQUISITE PROPOSITIONS CARRIED TO
FIRST APPROXIMATIONS;

WITH

THE CONSTRUCTION OF OPTICAL INSTRUMENTS.

For the Use of Junior University Students.

BY

RICHARD POTTER, A.M., F.C.P.S.,

LATE FELLOW OF QUEEN'S COLLEGE, CAMBRIDGE; LICENTIATE OF THE ROYAL COLLEGE
OF PHYSICIANS, LONDON; HONORARY MEMBER OF THE LITERARY AND
PHILOSOPHICAL SOCIETY OF ST. ANDREW'S;
LATE PROFESSOR OF NATURAL PHILOSOPHY AND ASTRONOMY
IN UNIVERSITY COLLEGE, LONDON.

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PREFACE.

THE author, in the present work, has adopted the plan of separating the more elementary from the higher parts of Geometrical Optics, so as to form, in the present First Part, an elementary treatise carried as far as can be done by taking the propositions to first approximations only. In a Second Part* he will treat of the problems requiring the differential calculus, the aberrations, oblique pencils, caustics, further corrections of some of the instruments, &c.

He hopes, by this subdivision of the science, to render the elementary part attainable, in a useful and effective manner, to the student who is not very far advanced in mathematics, and also to facilitate the comprehension of the whole to those who are more advanced, by presenting in the first instance to their attention the essential propositions in their more elementary form.

The author has followed the method used by the late Mr. Coddington, to whom the science is under so many obligations, in taking the rays of light positive each in their own directions, as is usual with respect to lines in the mechanical sciences, reserving the analytical rules for the cases where the subject will be treated as an analytical science. The treating a mixed geometrical science on imperfect analytical methods has led, he believes, to much misconception of the fundamental parts of the subject by the less advanced students; and if that method had the advantage of serving for an artificial memory in preparing for examinations, it had the overwhelming disadvantage of leaving incorrect ideas upon the actual applications of the science.

---

* Published January 1851.

The author hopes that he has met the views of other teachers of Natural Philosophy, whilst preparing a text-book for his own classes.

LONDON, JANUARY 1847.

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OPTICS.

PART I.

INTRODUCTION.

THE science of *Optics* is that, in which the properties and laws of *light* and *vision* are treated of.

It is subdivided into *Geometrical* Optics and *Physical* Optics: the former comprehending the application of certain simple determined properties or laws of light to the theory of optical instruments, and the structure of the eye; and the latter treating of the phenomena arising from properties which are connected with the more intimate nature of light, as interference, polarization, &c.

In the present *First* Part of Geometrical Optics, the theory of optical instruments will be discussed, by taking the propositions to first approximations only; and in the *Second* Part the propositions will be discussed to higher approximations, and the more corrected forms of some of the instruments treated of. This method involves the advantages of a more elementary treatise in the First Part, and a higher one in combining the two Parts.

Our whole perception and knowledge of the existence of light depends on the eye, as the organ of the sense of vision; and through it we receive the wonderful perception of the forms, colours, positions, &c. of distant objects, by the light coming from them. Every portion of space through which light traverses, is in Optics called a *medium*. Thus *a vacuum*, the *atmosphere*, transparent *liquids* and *solids*, are called media.

Bodies are *translucent* which transmit light, but in so disturbed a manner that the forms of objects at a distance beyond them cannot be distinguished; the term *transparency* being more correctly restricted to media through which the forms of distant objects can be seen.

Opaque bodies are such as *reflect* but do not *transmit* light. Few bodies are so opaque that they do not exhibit some degree of translucency when reduced to very thin laminæ. Thus several of the metals can be beaten into leaves so thin as to show considerable translucency, though in none has true transparency been established.

No dense medium is known which possesses *perfect* transparency; for independently of the light which is scattered in all directions, or dispersed, as it passes from one medium to another, and by which we perceive the presence of the bounding surface, the light which enters the medium is also subject to diminution from being *absorbed,* as well as dispersed. Coloured media absorb light of some colours and transmit light of other colours; a moderate thickness of some media being opaque for some colours and transparent for others: as, for instance, red glass and many red-coloured liquids transmit nearly all the red light falling on them, but no sensible part of *pure* yellow, green, blue, or violet light.

Rough surfaces of bodies scatter or disperse a large portion of the light falling on them, by which their peculiarities of figure, colour, &c. are seen by eyes situated in a variety of positions. *Smooth and polished* surfaces, on the contrary, reflect light according to a particular law, called *the Law of the reflexion of Light;* and the quantity so reflected is very different at different surfaces, being only about one-thirtieth when light falls perpendicularly upon a surface of common glass, about two-thirds in a perpendicular reflexion from ordinary silvered looking-glass, and a little more from highly polished speculum metal.

When light traverses a transparent medium, its direction is in a straight line from one point through which it passes, to any other, for it will pass along a straight tube between the two points, but not along a bent one, unless by being reflected. When it passes from a given point in one medium to another point in another medium of *different density*, falling obliquely upon their common surface, its path is made up of two straight lines, which meet in some angle at the common boundary of the two media. The abrupt change in the direction of light, when it passes from one medium to another of different density, is subject to a particular law, called *the Law of the refraction of Light.*

It is familiar to every one that the illumination is feebler as the distance from an origin of light is greater: this change of illuminating power is expressed in *the Law of the intensity of Light at different distances from a luminous body.*

The same *law of refraction* holds good for light of all colours; but the amount of refraction or bending at given surfaces is different for each colour; and in the passage of light through a prism, gives rise to the *chromatic* or *coloured* dispersion of light.

The three laws above named, and the properties of media in respect of chromatic dispersion, are the foundation of the principal propositions in *Geometrical Optics.*

In speaking of distinct portions of light, we continually use the terms, *a ray of light*, and *a pencil of rays*, or *a pencil of light*. By a *ray of light* we mean, the smallest portion of light which can be conceived; it is represented by the line or lines which constitute its path. By a *pencil of light* we mean, an assemblage of rays constituting a cylindrical or conical stream of light.

On account of the *physical* properties of light, we can never separate a pencil of the above forms *accurately* from the rest of the light which radiates from any luminous point; but the effect alluded to, taking place only near the boundaries of the pencil, will not sensibly disturb our geometrical results, unless the pencil be reduced to small dimensions: it, however, prevents our separating so small a pencil, that in delicate experiments we could consider it a *ray*. From what has just been said, it will be perceived that a knowledge of Physical Optics* is necessary in order to understand thoroughly the phenomena presented in many cases by optical instruments.

---

* See the Author's treatises, Part 1 and Part 2 of Physical Optics.

A stream of light is called a *converging* pencil when the rays converge to the vertex of the cone (called a *focus*), and a *diverging* pencil when they diverge from it. The axis of the cone in each case is called the axis of the pencil. When the stream consists of *parallel* rays, the pencil is called *cylindrical*; and the axis of the cylinder is called the axis of the pencil.

In *nature*, light diverges in all possible directions around each luminous point, so that our pencils are primarily *all diverging*; but when the luminous origin is very distant, as in the case of the heavenly bodies, the rays in every pencil we consider are sensibly parallel, and the results are those for *cylindrical* pencils. Though *converging* pencils do not exist originally in natural streams of light, they nevertheless are continually produced in optical instruments.

The various sources of light are divided into two classes: those of *self-luminous* origins, as the sun, the fixed stars, the flames of burning substances, the electric light, &c.; and those *secondary* origins of light, the surfaces of opaque bodies, which are visible only by the light they reflect, and are invisible in the dark, as the moon and planets, and opaque bodies generally.

As far as the propositions of Geometrical Optics are concerned, it is immaterial from which of these classes the light arises.

Light traverses the planetary spaces with a velocity of about 190,000 *miles per second of time*. This was first established by the Danish astronomer Rœmer, who found that the *appearance* of the eclipses of Jupiter's satellites occurred about sixteen minutes later than the calculated times when that planet and the earth were in opposite directions from the sun, and therefore were the most distant, the calculations being adapted to times when they were in the same direction from the sun, and therefore were nearest to each other. The difference of these distances is the diameter of the earth's orbit; so that we conclude that light comes from the sun in about eight minutes, giving a velocity of 192,500 miles per second, from the older observations, and about 183,470 miles per second, from the more recent observations.

Dr. Bradley afterwards discovered that the fixed stars have a change of apparent place during the earth's revolution round the sun, which arises from the composition of the velocity of light with the velocity of the earth in its orbit; this is called *the aberration* of the fixed stars. The velocity of light, deduced from observing the amount of this aberration, is but slightly different from the former.

We may consider the above as the velocity of light in a vacuum, whether it comes from a primary or secondary origin; but we find in Physical Optics that the velocity is *slower* in dense media, depending on their *refractive powers*.

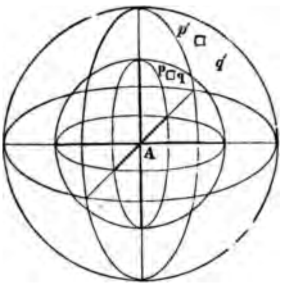
It is not necessary, in order that a point may remain continually visible, that the light must radiate from it in a continuous stream; for the impression of light on the eye is found to remain a sensible time, which probably differs with different eyes, being about one-seventh or one-eighth part of a second. This is shown by causing a piece of lighted charcoal, tied to a cord, to be whirled round in a circle in the dark. If the revolutions are about seven or eight in a second, the whole circle appears continuously luminous. During this interval, it will be seen that light would travel a distance of 25,000 miles.

ON THE THREE FUNDAMENTAL LAWS OF LIGHT.

DEFINITION. *The intensity* of the light falling on any surface is measured by the quantity of light, or the number of equally bright rays falling on a unit of area, supposing the intensity uniform.

Whilst we only require *relative* values of intensity, the above definition is sufficient, without knowing absolute measures depending on the assumption of some definite unit. Such a unit might, however, be found in the light given out by a unit of volume of a gas burned in a given manner.

LAW 1. *The intensity of the light at different distances from a luminous point is inversely as the square of the distance.*



Let A be the luminous point from which light radiates uniformly in all directions; pq a given *small* area at a distance D from A, and perpendicular to the line drawn from its center to A; $p'q'$ an equal area similarly situated at a distance D'.

Let I be the intensity at the distance D, and I' at the distance D'. If we suppose the surfaces of two spheres, whose radii are D and D' respectively, to be described through pq and $p'q'$, the whole radiant light will pass uniformly through their surfaces with the intensity at each proportional to the quantity of light on a unit of area.

$$\text{Or, } \frac{I}{I'} = \frac{\text{quantity of light falling on the area } pq}{\text{quantity of light falling on the area } p'q'}$$

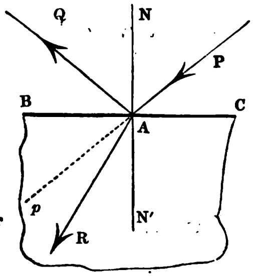
$$= \frac{\text{whole radiant light from } A \times \dfrac{\text{area of } pq}{\text{area of the spherical surface whose radius} = D}}{\text{whole radiant light from } A \times \dfrac{\text{area of } p'q'}{\text{area of the spherical surface whose radius} = D'}}$$

$$= \frac{4\pi D'^2}{4\pi D^2}$$

$$= \frac{D'^2}{D^2}$$

or, intensity at a distance D : intensity at a distance $D' :: \frac{1}{D^2} : \frac{1}{D'^2}$.

Definitions. When a ray of light falls upon any surface of a medium, the angle which it makes with the perpendicular to the surface at the point of incidence is called *the angle of incidence.*



The angle which the reflected ray makes with the same perpendicular is called the *angle of reflexion.*

The angle which the refracted ray within the medium makes with the same perpendicular produced, is called *the angle of refraction.*

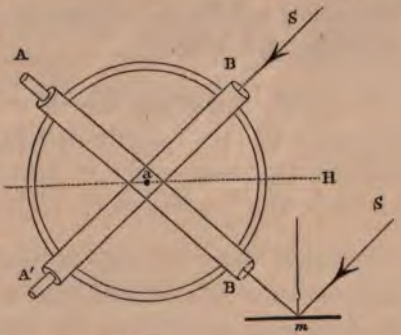
The angle which the reflected or refracted ray makes with the incident ray produced, is called *the angle of deviation*, or *deviation* of such ray.

If PA in the figure be a ray incident on the surface BAC of the medium, AQ the reflected ray, AR the refracted ray, NAN' the normal to the surface at A; then PAN is the angle of incidence, QAN the angle of reflexion, RAN' the angle of refraction; also, if we produce PA to p, RAp is the deviation of the refracted ray, and QAp is the deviation of the reflected ray.

Law 2. In the reflexion of light, *the incident ray, the normal to the reflecting surface at the point of incidence, and the reflected ray lie all in one plane, and the angle of reflexion is equal to the angle of incidence.*

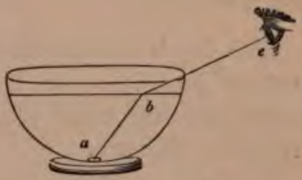
This law is easily verified experimentally; but the most severe tests of its accuracy are the observations, in astronomical observatories, with the mural circle, which are taken by reflexion in a trough of mercury.

Let AaB be the position of the telescope of the mural circle, when the image of a star is seen upon the horizontal wire in the focus of the eye-glass, by reflexion at the horizontal surface of the mercury at m; $A'aB'$ when the image is seen in the same position by light from the star directly. Then the angle AaA', through which the telescope has been turned, is twice the altitude $B'aH$ of the star; aH being a horizontal line.



The truth of this result depends on the perfect accuracy of the law of reflexion; and as the mural circle is the most accurate of all divided instruments, if the altitudes of the stars, obtained from direct observations, combined with observations by reflexion in mercury, had not agreed with those found by means of the vertical direction determined by the plumb-line, the error must have been detected.

LAW 3. In the refraction of light, *the incident ray, the normal to the refracting surface at the point of incidence, and the refracted ray lie all in one plane, and the sine of the angle of refraction bears a constant ratio to the sine of the angle of incidence; the ray in the dense medium lying nearer to the normal than in the rare medium.*



That the direction of light, in passing obliquely from one medium into another, is abruptly changed at the common surface of the media, and is further from the perpendicular in the rare medium, is easily shown experimentally by putting a piece of coin at the bottom of a basin, as at *a* in the figure, and then placing the eye as at *e*, so that the coin cannot be seen for the interposing edge of the basin; by pouring water into the basin, the coin becomes visible in the direction *e b*, the light from *a* following the broken path *a b e*.

By more accurate experiments the truth of the law is verified; and the applications of the law in the construction of optical instruments, giving results in accordance with facts in the most minute particulars, lead us to conclude its general truth.

The ratio which the sine of the angle of refraction bears to the sine of the angle of incidence is generally represented by the Greek letter μ, and is called the refractive index; or

$$\frac{\text{sine of angle of incidence}}{\text{sine of angle of refraction}} = \mu.$$

In the law of refraction it is meant, that whilst the two media in which the light moves are the same, the value of μ is constant, however the angle of incidence may vary; but μ has different values for different media, and generally is greater when the light passes into more dense or into inflammable media. It has also different values for different colours of light with the same media, giving rise thus to the chromatic dispersion.

The above is called the law of ordinary refraction, and holds good for glass, transparent liquids, and uncrystallized solids, and for many crystals; there are, however, many other crystallized bodies which have double refraction, a single ray in air giving rise to two refracted rays in the crystal. One of these follows, in some crystals, the law of ordinary refraction, and the other follows an extraordinary law of refraction; in other crystals both the refracted rays have extraordinary refraction. For the phenomena of double refraction the student is referred to Physical Optics.

When the light passes from a vacuum into a dense medium, μ is then called the *absolute refractive index*, and is always greater than unity. For some colours of light, μ is about $\frac{4}{3}$ for water, about $\frac{3}{2}$ for glass, and about 2·5 for diamond. A table of refractive indices will be found at the end of the volume. When light passes from one *dense* medium into *another*, the ratio of the sines of incidence and refraction is then called the *relative* refractive index of the media. If a ray of light passes *from* a dense medium into a vacuum, the refractive index is then $\frac{1}{\mu}$, and the law of refraction holds good with this value.

The refractive index for a vacuum into air is so nearly equal to unity, that we may, in ordinary questions, take the refraction from air into any dense medium as if from a vacuum into the medium, without any sensible inaccuracy.

It will be seen from what has been said, that the path of a ray will be the same whichever way it comes, both in cases of reflexion and refraction; so that in the solution of optical problems we may take the light as passing either one way or the other, along the path determined by the law of reflexion or refraction respectively.

The science of Optics is sometimes subdivided into *Catoptrics*, in which the reflexion of light is treated of; and *Dioptrics*, that in which the refraction of light is treated of.

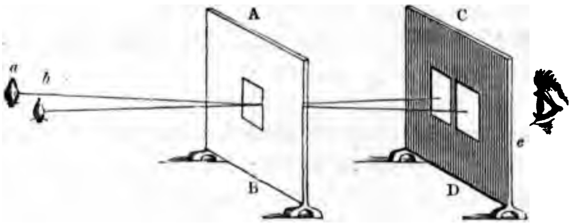
CHAPTER I.

ON PHOTOMETRY.

This branch of Optics is of great importance not only in determining the relative values of different sources of artificial light, but in comparing the effectiveness of different optical instruments, and in many problems of Physical Optics.

Three classes of instruments have been proposed as photometers, of which only one class strictly deserves the name; the other two, which depend on the heating and chemical powers which generally accompany light, cannot be considered as indicating invariably measures of that effect of light which is concerned in producing the sensation of vision. A reference to the judgment of the eye becomes, therefore, essential in a true photometer; and the part the eye has to perform is the determining when the illuminations produced from two different sources are *equal,* the indications of the instrument then giving the means of calculating the relative intensities required in the problem.

Article I. A simple and effective photometer may be constructed as in the figure; where *AB* is an opaque screen with a rectangular aperture in its center; and *CD* is another screen, set as nearly perpendicular to the direction of the lights as possible, of which a large aperture in its centre is covered with thin tissue-paper. The flames *a* and *b,* whose light is to be compared, will each illuminate a rectangular portion of the tissue-paper by



the light passing through the aperture in *AB*, as shown in the figure; and an eye placed as at *e*, behind the screen *CD*, will distinguish through the tissue-paper with considerable accuracy, when the two illuminations are equal, one of the flames being moved nearer or further away, until its illumination equals that of the other.

Let I_1 and I_2 be the intensities of the illuminations from *a* and *b* respectively at the distance *unity*; D_1 and D_2 respectively their distances from the screen *CD* when they illuminate the tissue-paper equally. We have, from Law 1,

$$\frac{\text{intensity at distance } D}{\text{intensity at distance } 1}=\frac{1}{D^2},$$

$$\therefore \text{ intensity from each} = \frac{I_1}{D_1^2}=\frac{I_2}{D_2^2}$$

$$\text{or, } \frac{I_1}{I_2}=\frac{D_1^2}{D_2^2};$$

or, the intensities which measure the values of the illuminating powers are directly as the squares of the distances from the screen.

In place of the screen *AB*, a straight rod may be set up, of which the two shadows may be received upon any white surface at *CD*. The eye being now in front of *CD* will see when the shadows are of equal intensity, but not with nearly the accuracy of the former method, as the shadows are surrounded by the double illumination, and the eye is not so favourably situated. In this method it will be seen that each flame shines upon the shadow of the rod cast by the other.

The same principle may be modified in many ways, so as to determine the intensities of lights; as, for instance, the portion which is transmitted by different transparent media, or reflected by various kinds of mirrors.

Many different forms of photometers, invented by Bouguer, and described in his 'Traité d'Optique,' will be found noticed in Dr. Priestley's 'History of Vision.' The most sensitive photometer is the one which was used by the author in his researches upon the reflective powers of metals and glass. (See 'Brewster's Edinburgh Journal of Science' for 1831.)

Art. 2. Prop. *To show how objects appear equally bright at all distances from the eye.*

The distance is supposed not to be so great that any sensible part of the light is lost by its passing through the air.

Each very small part of the surface of an object may be taken as a luminous point, and the intensity of the light from it will be inversely as the distance squared; but when the object is removed from the eye, the *apparent* linear dimensions become *less* directly as the distance becomes *greater*, or the apparent area of its surface becomes *inversely* proportional to the square of the distance; and the same number of luminous points being congregated upon this apparently smaller area, the number of such points upon a *given apparent area* is proportional to the distance squared.

Now, the apparent brightness of any surface is proportional to the intensity of the light at the eye from each luminous point, and to the number of luminous points in a given apparent area; that is, the apparent brightness varies as $\frac{1}{(Distance)^2}$ from the first cause, and as $(Distance)^2$ from the second, or it remains constant.

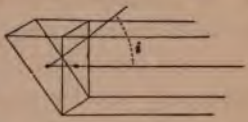
Art. 3. Prop. *If a small plane surface be illuminated directly by the light from another such surface, the illumination is proportional to the area of the latter × intensity of the light at a distance unity from each of its points ÷ the distance between the surfaces squared.*

For since the surfaces are both small, and the light falls directly, or perpendicularly, upon the illuminated one, the illumination must depend on that from each elementary part or luminous point, and on the number of such points or on the magnitude of the illuminating area. But the illumination from each point being $\frac{I}{D^2}$, and putting A for the illuminating area, expressed in terms of the number of elementary portions, which are each taken as luminous points, we have the whole illumination at the surface

$$=\frac{A\times I}{D^2}.$$

COR. The expression just found may be taken to represent the apparent brightness of the illuminated surface; and if the light is incident obliquely upon it, at an angle of incidence $=i$, the apparent brightness $=\frac{A\times I}{D^2}\cos. i$.

For the distance D being supposed to be great compared with the magnitude of the planes, we may suppose the light to form a pencil of parallel rays. The apparent brightness being proportional to the number of rays falling on a given area, we see by the annexed figure that the number falling on any area of the perpendicular surface is to the number on an equal area of the surface inclined to it at an angle i, as 1 to $\cos. i$.



Therefore the apparent brightness of the inclined surface

$$=\frac{A\times I}{D^2}\cos. i.$$

ART. 4. With respect to the intensity of the light radiating from a *luminous* surface at different inclinations, it has been found that such surfaces appear equally bright at all inclinations; but the number of luminous points in any given *apparent* area is proportional to secant i; hence we deduce that the intensity of the light radiating round *each* luminous point varies as $\frac{1}{\sec. i}$, or as $\cos. i$; i being, as before, the angle between the ray and the normal to the surface.

EXAMPLES.

Ex. 1. If an argand lamp gives an illumination upon a screen when 9 feet distant from it, equal to that from a wax-light when

3 feet from it, show that the light of the lamp is equivalent to that of nine such wax-lights.

Ex. 2. If a cubic inches of one kind of gas, burned at a distance of m feet from a screen, give an equal illumination to b cubic inches of another kind, burned in the same time, at a distance of n feet; required the relative values of the gases as sources of artificial light.

One cubic foot of the first $= \frac{b}{a} \cdot \frac{m^2}{n^2}$ of a cubic foot of the second.

Ex. 3. The illuminations from two flames upon a screen are the same when one of them is at a distance of 40 inches from it; but when a piece of window-glass is interposed between it and the screen, it requires to be drawn to a distance of 38 inches in order that the illuminations may be again equal: show that the glass transmits rather more than 90 rays of every 100 incident.

Ex. 4. If one-tenth of a beam of light is lost in passing through a plate of window-glass, how many rays of every 100 incident will emerge after passing through three such plates?

Ex. 5. In the experiment of article 1, if two lights at a give each separately equal illuminations on the screen with the light b at 40 inches; show that if the two be placed together near a, they will illuminate the screen equally with the light b when placed at 28·284 inches.

Ex. 6. If a mirror of speculum metal reflects seven-tenths of the light incident upon it, how many rays of every 100 incident would be effective in a telescope which has two such reflexions in its construction?

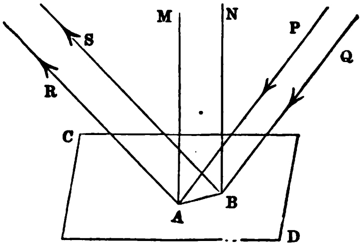
CHAPTER II.

ON THE REFLEXION OF LIGHT BY PLANE MIRRORS.

In the following propositions it will be shown that the effect of a plane mirror is to change the direction of a pencil falling upon it, without affecting its condition as to parallelism, divergence, or convergence, although only a portion of the incident light is reflected.

Art. 5. Prop. *When a pencil of parallel rays falls upon a plane mirror, the reflected pencil consists of parallel rays.*

Let PA, QB be any two of the parallel rays of the incident pencil, falling upon the plane mirror CD at the points A and B respectively. Let AM, BN be the normals to the mirror at A and B respectively, which are parallel; by Euclid, book xi. prop. 6.



Let the ray PA be reflected in the direction AR, and joining AB, let the plane RAB meet the plane QBN in the line BS; then BS is the direction of the ray QB after reflexion, and is parallel to RA. For, by the law of reflexion, the lines PA, AM, AR, are all in one plane, and

angle of reflexion MAR = angle of incidence MAP
= angle . . . . . . NBQ . . . by Euclid, book xi. prop. 10.

Also, plane QBS is parallel to plane PAR . . . by Euclid, book xi. prop. 15.

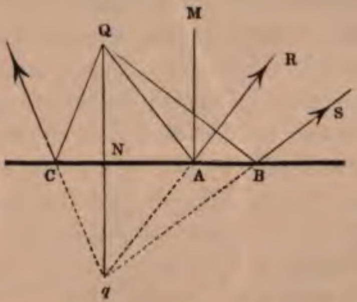
And line BS is parallel to line AR . . . by Euclid, book xi. prop. 16.

Therefore
$$\begin{aligned} \angle SBN &= \angle RAM \\ &= \angle PAM \\ &= \angle QBN \end{aligned}$$

And SB is the reflected ray corresponding to the incident ray QB.

The same holds good for any other rays, and the reflected pencil consists of parallel rays. It is easily seen that the distance of the rays is the same before and after reflexion, so that the transverse section of the pencil is unchanged.

ART. 6. PROP. *If a diverging or converging pencil is incident on a plane mirror, the focus of the reflected pencil is situated on the opposite side of the mirror to that of the incident pencil, and at an equal distance from it.*



First, let the pencil be a diverging one, and let Q be the focus of the incident rays, or point from which they diverge, in the figure. Let the line $CNAB$ represent a section of the surface of the mirror, draw the line QNq perpendicular to CAB, and make $qN=QN$; then q is the focus of the reflected rays. For let QA, QB, QC be any of the incident rays in the plane of the figure; draw the line AM perpendicular to CAB, and AR, making the angle $MAR =$ angle of incidence MAQ; then AR is the reflected ray. Join qA; the right-angled triangles QAN, qAN are equal by construction, and

$$\begin{aligned} \angle NqA &= \angle NQA \\ &= \text{alternate } \angle MAQ \\ &= \angle MAR \end{aligned}$$

$$\text{and} \quad \begin{aligned} \angle QAq + \angle QAR &= 2 . \angle QAN + 2 . \angle NQA \\ &= 2 \text{ right angles} \end{aligned}$$

$\therefore$ qA and AR are in one straight line.

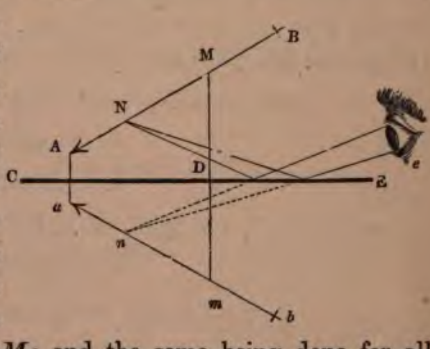
Similarly it is shown that the direction of every reflected ray is as if it had proceeded from q; or q is the focus of the reflected rays.

In this case, q is called a *virtual* focus; the term *real* focus being used in contradistinction, to indicate that the rays have actually passed through the point so called. As far as the optical properties of the rays are concerned, it is immaterial whether they proceed from a real or a virtual focus; and an eye, into which the reflected rays enter, perceives the point q as a *virtual* image of the luminous point Q.

It is evident that, if a pencil of rays *converging* to q had fallen upon the mirror, they would have been reflected to a *real* focus Q, and have then diverged again.

The ray QN, which falls perpendicularly upon the mirror, is reflected directly back again; and whenever we see a luminous point and its image in one straight line, that straight line is perpendicular to the mirror. This property is of great service in many optical experiments; for instance, if we want to determine whether the two surfaces of a plate of glass are accurately parallel, we do it most readily by placing a pin or needle directly before the pupil of the eye, and observing whether the two images of the light reflected by it, which are formed by the two surfaces of the glass plate, coincide in every way; and if they do, the surfaces at that part of the plate must be parallel.

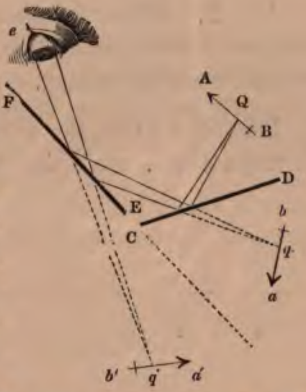
Art. 7. Prop. *To find the position and form of the image of an object placed before a plane mirror.*



Let the mirror be perpendicular to the plane of the paper, and intersect it in the line CDE. Let the arrow AB, in the plane of the paper, be the object. From any point M draw the perpendicular to the mirror MDm, and make $mD = MD$; then m is the virtual image of the point M; and the same being done for all the points in AB, we have a series of corresponding points forming a virtual image ab, situated as in the figure. This image is evidently of the same form and magnitude as the object, but inclined differently; the point *a*, the highest in the image, corresponding to the point *A*, the lowest in the object.

If the object were of any form whatever, we should find the image in the same manner to be of the same form and magnitude, at an equal distance from the mirror on the opposite side of it, but with its different parts inverted with respect to a given direction; as in the ordinary use of a looking-glass, we perceive that the right hand of the image corresponds to the left hand of the object. If an eye were situated at *e* in the figure, the small pencil which entered the pupil from any point *N* in the object would proceed as if from *n*, the corresponding point in the image.

ART. 8. PROP. *To find the form and position of the image when the light from an object has been reflected at each of two plane mirrors.*



Let *AB* be the object, *CD*, *EF* the two mirrors. Drawing perpendiculars from the various points in *AB*, and measuring equal distances on the opposite side of the mirror *CD*, we find the position of *ab* the image given by it. The light, after reflexion, proceeds as if it had come from an object at *ab*; and such part of it as falls on the mirror *EF* will be reflected accordingly, and proceed, after the second reflexion, as if it had come from a secondary image *a'b'*, whose position is determined by drawing perpendiculars upon *FE* produced, and measuring equal distances upon the opposite side of it.

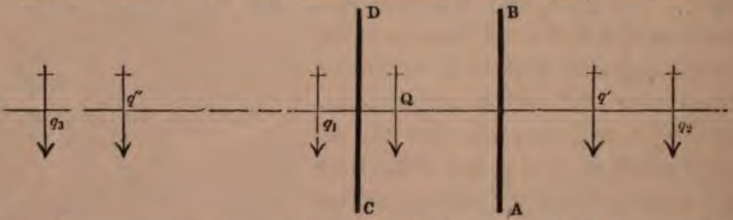
To trace the course of a pencil which enters an eye at *e* after reflexion at the two mirrors from any point *Q* in *AB*; let *q* be the corresponding point in *ab*, and *q'* that in *a'b'*: the light after reflexion by *EF* will enter the eye as if it came from *q'*, therefore drawing the lines *q'e* in the figure, they will represent the final course of the pencil: from the points where the lines meet the mirror *EF* draw lines to the point q, they will represent the course of the pencil after reflexion by *CD*: from the points where these lines meet the mirror *CD* draw lines to the point Q, and they will be the course of the rays from Q, which, after reflexion by each of the two mirrors, form the visual pencil by which the eye e sees q' in the secondary image $a'b'$.

It will be seen that the inversions of parts by each of the two mirrors now correct each other, and there is the same relative direction of parts to the eye in the secondary image $a'b'$ as in the object *AB*; that is, there is no inversion of upward and downward, or right hand and left hand. This principle is made use of in the *Camera lucida* of Dr. Wollaston, of which a description will be found in the chapter on instruments.

Cor. By proceeding in the same way, the positions of the series of images given by any number of mirrors may be found, and the course of a visual pencil traced.

Art. 9. *When two plane mirrors are placed with their reflecting surfaces towards each other and parallel, they form the experiment called the* endless gallery.

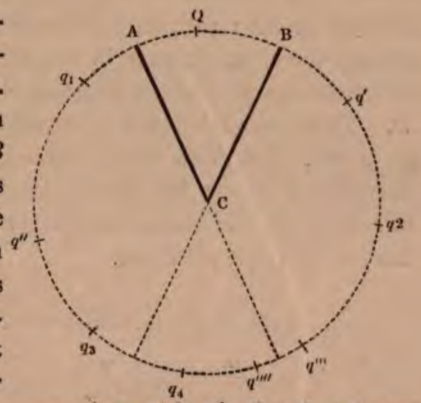
Let *AB*, *CD* be the parallel mirrors, Q the place of an object between them. We find the positions of the images as in the



preceding propositions. If q_1 be the image of Q formed by the mirror *CD*, we shall have q_2 an image of q_1 formed by the mirror *AB*, or secondary image of Q, as in the figure, and q_3 an image of q_2 formed by the mirror *CD*, or tertiary image of Q, &c. In the same way there will be formed a series of images, q', q'', &c., commencing with q', the image of Q formed by the mirror *AB*.

To an eye placed between the mirrors, the succession of images will be seen as described; and if the mirrors were perfectly plane, and reflected all the light incident upon them, the number would be infinite.

ART. 10. *To find the arrangement of the images formed by two plane mirrors inclined to each other as in the* kaleidoscope.

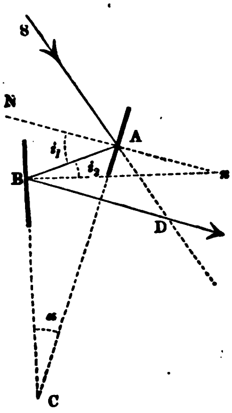


Let AC, BC be the perpendicular sections of the mirrors by the plane of the paper. Let Q be the position of an object within the angle ACB formed by the reflecting faces of the mirrors, and describe with center C a circle through Q; the two series of images of Q, viz. q_1, q_2, q_3, q_4, commencing with q_1, the first image formed by the mirror AC; and q', q'', q''', q'''', commencing with q', the first image formed by the mirror BC, will be all in the circumference of the circle. For since q_1 is determined by drawing a perpendicular to the mirror AC, and measuring an equal distance on the opposite side, q_1 is equally distant from C with Q; in the same way q_2 is is equally distant from C with q_1 and Q, and so with all the other images, or they are in the circumference of the circle drawn through Q with center C.

When we arrive at an image, as q_4 or q'''' in the figure, falling within the angle formed by the directions of the mirrors *produced*, that image is the last of its series; for whichever mirror it be formed by, since it lies at the back of both of them, no light proceeding as from it can fall on the face of the other mirror.

If the angle ACB be an aliquot part of 180°, the images of every point in the arc AQB will occur the same number of times, and the whole series will be symmetrical. The proof of this is long, but may be found in several optical treatises.

Art. 11. Prop. *To find the deviation of a ray of light after being reflected at each of two mirrors, inclined at a given angle, in a plane perpendicular to their intersection.*



Let the directions of the mirrors at A and B when produced meet at C, and let angle $ACB=\alpha$. Let SA be a ray of light incident on the first mirror at A, and be reflected in AB; and, after reflexion at the second mirror at B, pass in the direction BD, meeting the original direction SAD in D. Then the angle ADB is the deviation; let it $=d$. Let NAn be the normal to the first mirror at A; Bn that to the second mirror at B: the angle AnB between the normals = angle $ACB=\alpha$.

Let i_1 be the angle of incidence on the first mirror.
„ i_2 „ „ second „

$$\angle NAB = \angle ABn + \angle AnB$$

$$\text{or, } i_1 = i_2 + \alpha \quad \therefore \alpha = i_1 - i_2$$

$$\text{also, } \angle BAD + \angle ABD + \angle ADB = 180^\circ$$

$$\text{or, } 2(90^\circ - i_1) + 2i_2 + d = 180^\circ$$

$$\therefore d = 2(i_1 - i_2) = 2\alpha$$

or the deviation for every ray, after the two reflexions, equals twice the angle between the mirrors.

This property is made the principle on which Hadley's sextant and reflecting circles, for taking astronomical observations, are constructed, as will be seen in the chapter on instruments.

EXAMPLES.

Ex. 1. Show that when a ray from any given luminous point is reflected at a plane mirror, and passes through another given point, it traverses a shorter distance than it would have done if reflected by the mirror in any other manner.

Ex. 2. Show that persons standing at any distance from a plane vertical mirror may see the entire height of the image of themselves, if the mirror, at a proper elevation, be of half their height.

Ex. 3. When a drawing is made of an object, and its image reflected in a mirror, show how the properties of perspective affect the apparent magnitudes and positions.

Ex. 4. Trace the course of the visual pencil, when an image of an object is seen by reflexion in three plane mirrors.

Ex. 5. A star, whose altitude is n°, is to be observed by means of its image seen in a small vessel of mercury placed on the ground, by a person whose eye is m feet above the level of the mercury; show that he must stand at a distance from it of $m \cot. n^\circ$ feet.

Ex. 6. An object is situated n inches from a plane mirror, which, being turned through an angle of 60°, the positions of the object and the images in the two cases form an equilateral triangle; show that the distance of the point about which the mirror was turned was $\frac{2n}{\sqrt{3}}$ inches from the object.

Ex. 7. When a kaleidoscope is formed with three equal plane mirrors, having their perpendicular sections everywhere an equilateral triangle, what is the appearance in the field of view?

CHAPTER III.

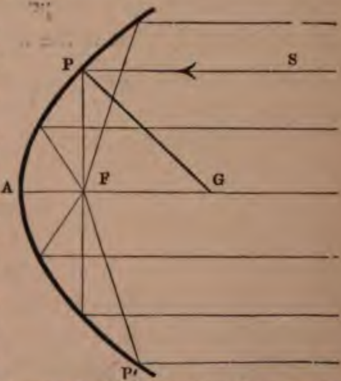
ON THE REFLEXION OF LIGHT BY CURVED MIRRORS.

When a pencil of light is incident upon a curved mirror, each ray of the pencil is reflected according to the ordinary law; and if we can draw the series of normals for the different points of the reflecting surface, we can determine the directions in which the various rays are reflected, as in plane mirrors.

In the following propositions the reflecting surfaces will be considered only of the forms, the sphere, and the paraboloid ellipsoid and hyperboloid of revolution; the three latter being formed by the revolution of the conical parabola about its axis, and the ellipse and hyperbola, about their major axes; because in these the mode of drawing the normal is sufficiently simple, and they are the only surfaces of importance in the theory of catoptrical instruments.

A mirror is called concave or convex when the reflecting surface is concave or convex respectively.

Art. 12. Prop. *If a pencil of parallel rays be incident on a concave parabolic reflector parallel to its axis, the whole will be reflected to the focus of the generating parabola.*

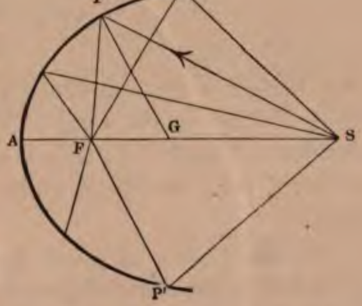


Let PAP' be a parabola, which is the section of the mirror by the plane of the paper passing through the axis AFG, of which A is the vertex and F the focus. Let SP be any one of the rays of the pencil, parallel to AFG, and incident at P; join FP, and let PG be the normal at P. Then, as shown in treatises on the conic sections, PG makes equal angles with SP and the line FP; and the angle of incidence being SPG, the angle

FPG is the angle of reflexion, and *PF* is the reflected ray. Similarly every other of the reflected rays passes through *F*.

Reciprocally, if *F* be a luminous point, every ray radiating from it will be reflected parallel to the axis of the parabola; and, the reflected pencil consisting of parallel rays, the light will be of equal intensity at all distances from the mirror.

ART. 13. PROP. *If a pencil of rays diverge from a luminous point in one of the foci of a concave elliptic reflector, they will be reflected to the other focus of the generating ellipse.*



Let *PAP'* be the ellipse which is the section of the mirror by the plane of the paper passing through the axis *AFS*, where *A* is one extremity of the major axis, and *F* and *S* the foci of the ellipse.

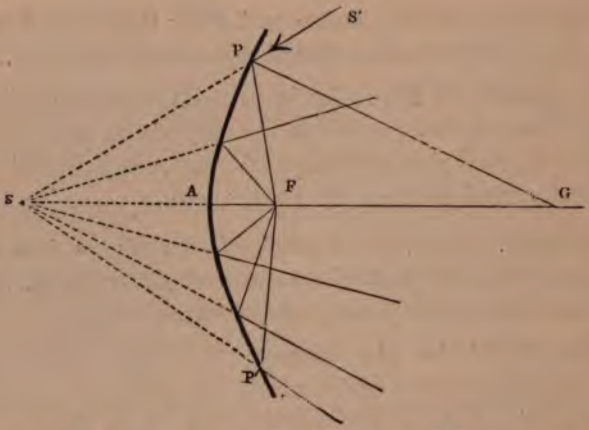
If *S* be the position of a luminous point, *SP* any one of the rays from it incident on the mirror at *P*, and we draw *PG* the normal and join *FP*; then, by conic sections, the angle *SPG*=angle *FPG*, and the former being the angle of incidence, the latter will be the angle of reflexion; therefore *PF* is the reflected ray. Similarly, it is shown that every other ray will, after reflexion, pass through *F*, or *F* is the focus of the reflected rays.

The same proof applies to the case when *F* is the position of the luminous point or focus of the incident rays, *S* being then the focus of the reflected rays.

ART. 14. PROP. *If a pencil of incident rays converge towards the exterior focus of a concave hyperbolic reflector, they will be reflected to the other focus of the generating hyperbola.*

Let *PAP'* be the generating hyperbola, *SAFG* the axis of the mirror, of which *A* is the vertex, and *S* and *F* the foci of the hyperbola.

Let *S* be the focus to which the incident pencil converges, *S'P* a



ray incident at P; draw the normal PG and join FP; by conic sections PG bisects the angle FPS'; and since $S'PG$ is the angle of incidence, therefore FPG is the angle of reflexion, and FP is the reflected ray. Similarly, every other ray of the pencil is reflected to F.

The same mode of proof shows, that if a luminous point were placed at F, the reflected pencil would diverge from S.

As the converse of the three preceding propositions, the following will be easily demonstrated in the same way.

Art. 15. Prop. *If a pencil of parallel rays be incident on a* convex *parabolic mirror, parallel to the axis, the reflected pencil will* diverge *from the focus of the generating parabola.*

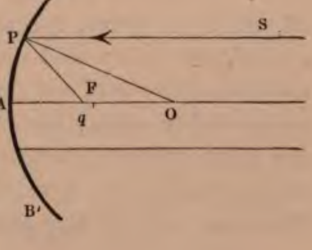
Art. 16. Prop. *If a pencil of rays converging towards one of the foci be incident on a* convex *elliptic mirror, the reflected pencil will* diverge *from the other focus.*

Art. 17. Prop. *If a pencil of rays diverging from the exterior focus be incident on a* convex *hyperbolic mirror, the reflected pencil will* diverge *from the other focus.*

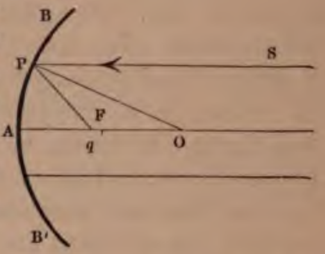
In the preceding Articles, the reflexion was accurately to or from a point as a real or virtual focus, however large the incident pencil might be; but parabolic, elliptic, and hyperbolic mirrors are very difficult to make, compared with spherical mirrors, and hence it is generally assumed that the mirrors of reflecting telescopes and microscopes are not so large as to preclude the use of a spherical figure.

The metal called speculum metal, which is used to form telescopic mirrors, is a compound of about 14½ parts tin and 32 parts copper, with sometimes an addition of a small quantity of arsenic or zinc. This alloy is very hard and brittle, and takes a high polish. The speculum or mirror, having been first cast in a mould of sand to nearly the required curvature, is then ground by rubbing in all directions with emery-powder and water on a tool of proper curvature, made of some softer metal, until the tool and speculum are of the same curvature as near as can be procured by using finer and finer emery. When a very accurate figure is required, the speculum has then to be worked on a tool formed of hones until the two coincide accurately, as seen by the fine lines on the face of the speculum; when this takes place, the surfaces of the hones and speculum must be truly spherical, as spherical and plane surfaces only can be worked upon each other in all directions and always coincide. To procure plane surfaces, three surfaces must be worked together alternately, until they coincide when taken any two together. When the speculum has been brought to a fine and accurate face, it is polished on a bed made with a mixture of purified pitch and rosin, with water and polishing powder, which is a fine calcined oxide of iron or tin. See, for further particulars of this delicate art, 'Brewster's Edinburgh Journal of Science' for 1831, and the articles "Grinding" and "Speculum" in 'Rees's Cyclopædia:' for a method of reducing a spherical mirror to an ellipsoidal, paraboloidal, or hyperboloidal figure, see 'Brewster's Edinburgh Journal of Science' for 1832.

Art. 18. *To find the focus of the reflected rays when a small pencil of parallel rays is incident directly on a concave spherical mirror.*



Let BAB' be the section of the spherical mirror, A the vertex or center of the aperture, O the center of the curvature. The line AO is called the axis of the mirror. Since the pencil is incident directly upon the mirror, the axis of the pencil will be in OA; and the ray which coincides with OA will be reflected directly back again; therefore the focus of the reflected rays must be in this line.

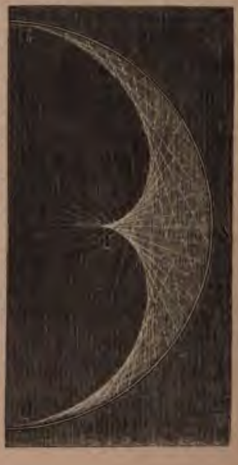


Let SP be any other of the parallel rays; join OP, and draw the line Pq, making the angle qPO = the angle of incidence SPO; then Pq is the reflected ray. But angle SPO equals alternate angle POq: therefore the triangle qPO is isosceles, and Pq equals qO.

When the incident pencil is indefinitely small, or P indefinitely near to A, the point q bisects the distance AO; for then Pq and Oq are each indefinitely nearly equal to half of AO. Let this position of q be F; then F is called the principal focus of the mirror, and AF the principal focal length, or generally the focal length, and it equals half the radius.

When the distance AP is not small, qO and qP are greater than half OP or OA, or q lies between F and A. The distance Fq is called the aberration of the ray, and it increases nearly as the square of AP, as is shown in Part II.

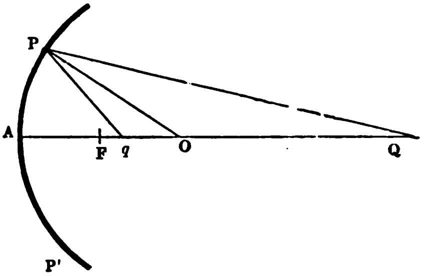
When the pencil is large, the continual intersection of the consecutive rays forms a curve as CFC in the figure, having a cusp at F, where the greatest condensation of light takes place. The curve CFC is called a *caustic curve*, and may be easily seen by placing a bright metal ring on a sheet of paper in the sun's light. When the incident rays are parallel, as in the figure, the caustic is an epicycloid, or curve traced out by a point in the circumference of one circle whilst it rolls upon another circle; the radius of the rolling circle being $\frac{1}{4}$ that of the mirror, and of the fixed circle $\frac{1}{2}$ of it, in this case. When the rays diverge from a point in the



circumference of the section of the mirror, the caustic is also an epicycloid; but in other cases it is not a determined curve.

ART. 19. PROP. *To find the focus of the reflected rays, when a small pencil of diverging rays is incident directly on a concave spherical mirror.*

Let PAP' be a section of the mirror, A the center of the aperture, F the principal focus, O the center of the curvature, and Q the focus of the incident rays.



The ray QOA passing through O, and falling perpendicularly upon the mirror at A, will be reflected directly back again. Let QP be any other ray of the incident pencil, and qP the reflected ray. Draw the radius OP which bisects the angle QPq; therefore, by Euclid, book vi. prop. 3, we have

$$QO : qO :: QP : qP.$$

When the pencil, for a first approximation, is very small, $QP = QA$, and $qP = qA$, very nearly.

$$\therefore \frac{QO}{QA} = \frac{qO}{qA} \text{ nearly.}$$

Let $QA = u$, $qA = v$, $AF =$ the principal focal length $= f$, $AO =$ radius of the mirror $= r$.

Then $f = \frac{r}{2}$, and the above equation becomes

$$\frac{u-r}{u} = \frac{r-v}{v}$$

Dividing by r, we have

$$\frac{1}{r} - \frac{1}{u} = \frac{1}{v} - \frac{1}{r}$$

$$\text{or, } \frac{1}{v} = \frac{2}{r} - \frac{1}{u}$$

$$= \frac{1}{f} - \frac{1}{u}$$

This expression gives us Aq or v when f and u are known. If the rays had diverged from q, it is evident, from the nature of the proof, that they would have been reflected to Q, and hence these points are called *conjugate foci.*

If we examine the figure, we see that if Q be moved further away from A, q approaches A; and when Q is at an infinite distance, or the incident rays are parallel, then q coincides with F: if Q approaches A, q recedes from it, and they meet at O; for then every ray, being incident perpendicularly upon the mirror, will be reflected directly back again: if Q lies between O and F, q lies beyond O from A; and when Q comes to F, q is at an infinite distance, and the reflected rays are parallel: if Q lies between A and F, the reflected rays diverge, and q lies on the opposite side of A; and when Q comes to A, q coincides with it again; if Q lies on the opposite side of A, or the incident pencil is convergent, q lies between A and F.

The relative positions of Q and q are easily deduced from the analytical relation, $\frac{1}{v} = \frac{1}{f} - \frac{1}{u}$: the quantities which are here considered positive being measured from A towards the right hand, must be taken negative when measured in the opposite direction.

Thus, if $u = \infty$ then $\frac{1}{v} = \frac{1}{f}$ and $v = f$

or, q coincides with F.

„ $u > r$ .... $\frac{1}{v} > \frac{1}{r} < \frac{1}{f}$ ... $v > f < r$

or, q lies between F and O.

„ $u = r$ .... $\frac{1}{v} = \frac{1}{r}$ ... $v = r$

or, q coincides with Q at O.

Thus, if $u<r>f$ then $\frac{1}{v}<\frac{1}{r}$ and $v>r$

or, q lies beyond O from A.

„ $u=f$ .... $\frac{1}{v}=0$ ... $v=\infty$

or, q is at an infinite distance.

„ $u<f$ .... $\frac{1}{v}$ is $-^{\text{ve}}$ ... v is $-^{\text{ve}}$

or, q lies to the left hand of A.

„ $u=0$ .... $\frac{1}{v}=-\infty$ ... $v=0$

or, q coincides with Q at A.

„ $u-^{\text{ve}}$ .... $\frac{1}{v}$ is $+^{\text{ve}}>\frac{1}{f}$ ... $v+^{\text{ve}}<f$

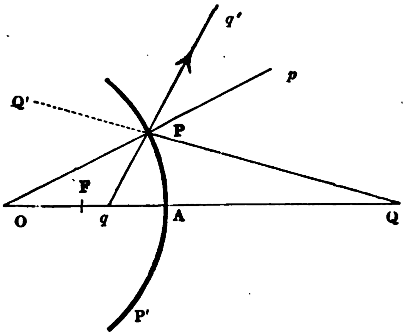
or, q lies between A and F.

„ $u=-\infty$ .... $\frac{1}{v}=\frac{1}{f}$ ... $v=f$

or, q coincides again with F.

These various positions of the conjugate foci are easily verified by experiments.

Art. 20. Prop. *To find the focus of the reflected rays when a small pencil of diverging rays is incident directly on a convex spherical mirror.*



Let PAP' be the mirror, of which A is the center of the aperture, and O the center of the curvature. Let Q be the focus of the incident rays, QA the ray which falls perpendicularly on the mirror, and is reflected directly back again, or in the line OAQ. Let QP be any other ray of the pencil; draw the radius OP, and produce it to p; draw the line Pq', making the angle $q'Pp=$ angle of incidence QPp; then Pq' is the reflected ray: let it, when produced, meet the axis of the mirror OA in q, which is the virtual focus of the reflected rays.

If we produce the direction of QP to Q', the line OP bisects the angle $Q'Pq$, the exterior angle of the triangle QPq; therefore, by Euclid, book vi. prop. A,

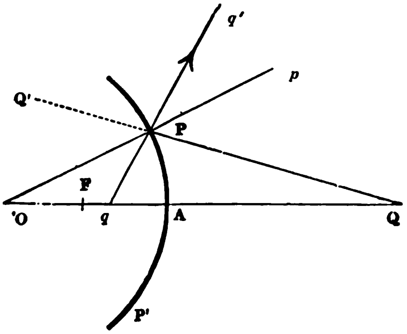
$$QO : qO :: QP : qP.$$

In a first approximation we suppose the pencil very small, and

$$QP = QA, \quad qP = qA, \text{ very nearly.}$$

$$\therefore \frac{QO}{QA} = \frac{qO}{qA} \text{ very nearly.}$$

The change of the relative positions of the conjugate foci Q and q may be traced geometrically, as in the last Article, and will be found to accord with the results of the analytical discussion below.



Let $QA = u$, $qA = v$, $OA = r$, $OF = AF = f = \frac{r}{2}$, then the last equation becomes

$$\frac{u+r}{u} = \frac{r-v}{v}$$

or, dividing by r, $$\frac{1}{r} + \frac{1}{u} = \frac{1}{v} - \frac{1}{r}$$

whence $$\frac{1}{v} = \frac{2}{r} + \frac{1}{u}$$

$$= \frac{1}{f} + \frac{1}{u}$$

From which equation v is found when f and u are given; and therefore the position of q, the virtual focus of the reflected rays, is known.

To trace the corresponding positions of the conjugate foci,

if $u=0$ then $\frac{1}{v}=\infty$ and $v=0$

or, Q and q coincide at A.

„ $+u<\infty$ $\ldots\ldots\frac{1}{v}>\frac{1}{f}$ $\ldots v<f$

or, q lies between A and F.

„ $u=\infty$ $\ldots\ldots\frac{1}{v}=\frac{1}{f}$ $\ldots v=f$

or, q coincides with F.

„ u is $-^{\text{ve}}>r\ldots\ldots\frac{1}{v}>\frac{1}{r}<\frac{1}{f}\ldots v<r>f$

or, q lies between F and O.

„ $-u=r$ $\ldots\ldots\frac{1}{v}=\frac{1}{r}$ $\ldots v=r$

or, q coincides with Q at O.

„ $-u<r>f\ldots\ldots\frac{1}{v}<\frac{1}{r}$ $\ldots v>r$

or, q lies beyond O from A.

„ $-u=f$ $\ldots\ldots\frac{1}{v}=0$ $\ldots v=\infty$

or, q is at infinite distance.

„ $-u<f$ $\ldots\ldots\frac{1}{v}$ is $-^{\text{ve}}$ $\ldots v$ is $-^{\text{ve}}$

or, q is on the right-hand side of A.

„ $-u=0$ $\ldots\ldots\frac{1}{v}=-\infty$ $\ldots v=0$

or, q and Q coincide at A again.

In the two preceding propositions a diverging pencil has been taken for a standard case, as incident on a concave and a convex mirror; but any case whatever being taken, and the corresponding formula being investigated, all the other cases can be deduced from it by changing the signs of the various lines when they have to be measured in a contrary direction.

Ex. 1. The formula for a concave mirror and a diverging pencil being $\frac{1}{v}=\frac{1}{f}-\frac{1}{u}$; to show that for a diverging pencil and a convex mirror we have $\frac{1}{v}=\frac{1}{f}+\frac{1}{u}$.

Since the focal length of the convex mirror is negative, if that of the concave is positive, therefore in $\frac{1}{v}=\frac{1}{f}-\frac{1}{u}$ we put $-f$ for f, and have

$$\frac{1}{v}=-\left(\frac{1}{f}+\frac{1}{u}\right)$$

which shews us that v must be negative; therefore, putting $-v$ for v, and changing the signs on both sides, we have

$$\frac{1}{v}=\frac{1}{f}+\frac{1}{u}$$

Ex. 2. To deduce the case of a plane mirror from the formula for a concave mirror.

In a plane mirror we must consider the radius as infinite, and $f=\infty$, or $\frac{1}{f}=0$

$$\therefore\ \frac{1}{v}=-\frac{1}{u} \text{ or } v=-u$$

or q is at an equal distance on the opposite side of the mirror to Q.

ART. 21. PROP. *To prove that in the reflexion of a pencil at a spherical mirror, the focal length is a mean proportional between the distances of the conjugate foci from the principal focus.*

Taking the expression $\frac{QO}{QA}=\frac{qO}{qA}$, as investigated in Articles 19 and 20, we have

$$\frac{u\mp r}{u}=\frac{r-v}{v}$$

$$\therefore\ \frac{u\mp r}{u}\pm 1=\frac{r-v}{v}\pm 1$$

bringing to a common denominator on each side, and dividing, we find

$$\frac{2u\mp r}{\mp r}=\frac{r}{r-2v}$$

or, $$\frac{u\mp\frac{r}{2}}{\mp\frac{r}{2}}=\frac{\frac{r}{2}}{\frac{r}{2}-v}$$

or, $$\frac{FQ}{FA}=\frac{FA}{Fq}$$

whence $$FQ\times Fq=FA^2=FO^2.$$

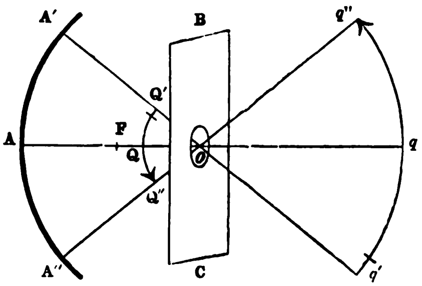
If the first formula were adapted to any other case, we should find the same result; and since the conjugate foci always lie on the same side of the principal focus, the above expression suffices to find either of them for a given mirror when the other is known.

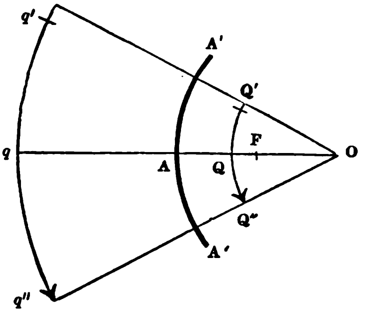
We have supposed the incident pencil to be small in the three preceding propositions; when it is large, the reflected pencil is not brought *very nearly* to a focus, but the rays which are incident at a distance from the axis of the pencil have an aberration which is too large to be neglected, and which prevents the use of such mirrors where an accurate image of a luminous point is indispensable.

ON THE FORMATION OF IMAGES BY SPHERICAL MIRRORS.

ART. 22. PROP. *To shew that if an object before a spherical mirror be a circular concentric arc, then the image formed by small direct pencils will be a similar concentric circular arc.*

First, let the mirror be *concave*, as $A'AA''$ in the figures; let O be the center, and $Q'QQ''$ the object, which is a circular arc with center O, and which lies between the principal focus F and O in this figure, and between A and F in the next.





Since the conjugate foci lie always on the ray which is perpendicular to the mirror, on the same side of *F*, and that $FQ \times Fq = FO^2$; if we draw lines from the center *O* through any points as *Q'*, *Q*, *Q''* in the object, we shall have the conjugate foci at *q'*, *q*, *q''* in the same lines as in the figures; and since the object is everywhere equally distant from *O*, therefore the image will be so also, and similar to the object.

In the first figure the image is *real* and *inverted*; in the second it is *virtual* and *erect*; in both it is larger than the object, or is *magnified* in the proportion

$$\frac{q'q}{Q'Q}=\frac{Oq}{OQ}=\frac{qA}{QA} \qquad \text{(Arts. 19 and 20)}$$

or the magnitudes of the image and object are proportional to their distances from the mirror respectively.

If *q'qq''* had been an object in the first figure, *Q'QQ''* would have been its *inverted*, *real*, and *diminished* image; and if the object be at an indefinitely great distance, the image is at the principal focus.

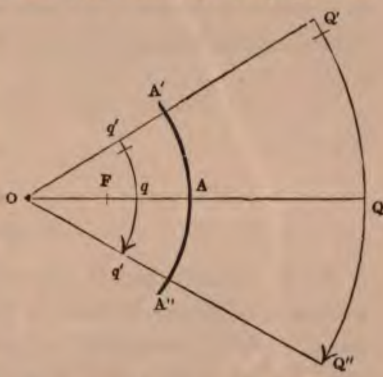
So also if a series of converging pencils fall upon the second mirror, so that they would form an image *q'qq''* if the mirror did not interpose, they will form a *real* image *Q'QQ''*.

By placing an opaque screen, as *BC* in the first figure, so that a small aperture in its center is at the center of the mirror, every pencil, limited by the aperture, will be *small* and *direct*; and therefore every point in the image will be *distinct*; and being *real* can be received on a screen of the proper curvature.

Small portions of the object may be considered as straight lines, and the corresponding portions of the image will be so also; but if a large object were straight, we see that the image would be curved like that of a circular arc; and it is found to be always one of the conic sections. See PART II. page 42.

If an eye be placed near O in the second figure, every pencil passing through the pupil will be small and reflected nearly directly at the mirror; and an eye so situated will see a *distinct*, *virtual*, *erect*, and *magnified* image. If the eye be in any other position, the parts of the image formed by oblique pencils will not be so distinct as those which are formed by direct ones.

Secondly, let the mirror $A'AA''$ be convex, O its center, $Q'QQ''$ the object; then we find the foci which are conjugate to the various points in the object, to form an image as $q'qq''$ in the figure; which will be always *virtual*, *erect*, and *diminished*, and will be seen with its parts of greater or less distinctness by an eye in front of the mirror, as the reflected visual pencils are more or less nearly direct.



If a series of converging pencils tended to form an image at $q'qq''$, they would form after reflexion a real image at $Q'QQ''$.

We have supposed the foci of the reflected rays to be those for *small, direct* pencils; if the pencils, although small, were oblique, then a fresh source of indistinctness in the image would be introduced, which is called *confusion*, depending on the degree of obliquity, and distinct from the aberration which depends on the figure of the mirror. See PART II. page 46.

We see that although a mirror might have the requisite figure, as a parabola, ellipse, or hyperbola, to reflect a large pencil from *one* point of an object accurately to or from a focus, yet for other points the incident pencils would be oblique, and the image would have more or less of confusion, for parts which were distant from the focus of accurate reflexion.

We see also that the accurate similarity of the image to the object depended on the form of the object being an arc concentric with the mirror; if the object had been a straight line perpendicular to the axis of the mirror, the centers of the object and image would have been similar, but distant parts would have been not only differently situated, arising from the curvature of the image, but the magnitudes of corresponding small parts would have borne a continually changing ratio to each other: this is called *distortion.*

The errors which therefore require to be remedied as much as possible in the use of mirrors in optical instruments, are *aberration, confusion, curvature,* and *distortion.*

The same errors affect the use of lenses, but in different degrees.

EXAMPLES.

Ex. 1. Shew that if the axis of a small concave spherical mirror be directed towards a planet, the linear diameter of the image equals the apparent angular diameter of the planet multiplied into the focal length, and the image is inverted.

Ex. 2. When the moon's apparent diameter is half a degree, shew that her image formed by a concave mirror of 100 inches radius is .4363 inch diameter.

Ex. 3. Show that if a small object be placed at the center of a small concave spherical mirror, the image will be equal in magnitude to the object, but inverted.

Ex. 4. An object being placed before a spherical mirror, shew its various positions, when the image is *real, virtual, erect, inverted, magnified,* or *diminished,* respectively.

Ex. 5. A pencil of parallel rays falling on a concave spherical mirror parallel to its axis, what is the aperture of the mirror when the rays falling near the edge of the mirror are reflected to the center of the aperture?

Ex. 6. If a parabolic burning speculum be placed directly in the radiant light and heat from the sun, shew that the heat in the focus is proportional to the square of the radius of the aperture of the speculum.

Ex. 7. If a series of small plane mirrors are to be combined to produce a burning mirror by the sun's heat at a given distance, how must they be arranged?

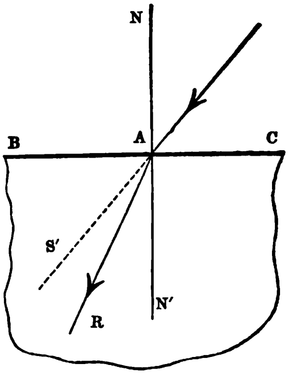
CHAPTER IV.

ON THE REFRACTION OF LIGHT BY MEDIA BOUNDED BY PLANE SURFACES.

It was explained, in the Introductory Chapter, on the law of refraction, that for homogeneous light, or whilst the rays remain of one colour, the index of refraction for a ray passing from one medium to another is constant, but has different values for differently coloured rays.

We commence with the propositions for homogeneous light, and afterwards proceed to those which relate to light of mixed colours.

Art. 23. Prop. *When a ray of light passes from a vacuum into a refracting medium, the angle of deviation increases as the angle of incidence increases.*



Let BAC be the surface of the medium, SAS' the direction of the ray incident at A, and refracted in the direction AR. Then if NAN' be the normal at A, SAN is the angle of incidence, RAN' is the angle of refraction, and $S'AR$ the angle of deviation.

Let $\angle SAN=i$, $\angle RAN'=i'$, $\angle S'AR=d$.

By the law of refraction $\frac{\sin i}{\sin i'}=\mu$, from which we see that the angles i and i' increase together.

Also, $d=i-i'$

$$\text{and} \qquad \frac{\sin i}{\sin i'}\mp 1=\mu\mp 1=\frac{\sin i\mp\sin i'}{\sin i'}$$

$$\therefore\ \frac{\sin i-\sin i'}{\sin i+\sin i'}=\frac{\mu-1}{\mu+1}=\frac{\tan\left(\frac{i-i'}{2}\right)}{\tan\left(\frac{i+i'}{2}\right)}\quad\text{(by trigonometry)}$$

$$\text{whence }\tan\frac{d}{2}=\tan\left(\frac{i-i'}{2}\right)=\frac{\mu-1}{\mu+1}\tan\left(\frac{i+i'}{2}\right)$$

and $\frac{i+i'}{2}$ is always less than 90°, and $\tan\left(\frac{i+i'}{2}\right)$ increases with i and i'; $\therefore$ $\tan\left(\frac{i-i'}{2}\right)$ and $i-i'$ or d, increase with i and i'.

When the angle of incidence is so small that we may put i for $\sin i$, and i' for $\sin i'$, using the circular measures of the angles, we have

$$i'=\frac{i}{\mu},\qquad\text{and } i-i'=\frac{\mu-1}{\mu}i=d$$

taking $\mu=\frac{4}{3}$ for water, we have $d=\frac{i}{4}$, when i is small.

. . . $\mu=\frac{3}{2}$ for glass, . . . $d=\frac{i}{3}$ . . . . . .

. . . $\mu=\frac{5}{2}$ for diamond, . . . $d=\frac{3i}{5}$ . . . . . .

ART. 24. PROP. *To find the value of* the critical angle, *or the limit of emergence, when a ray of light passes out of a dense medium into a vacuum.*

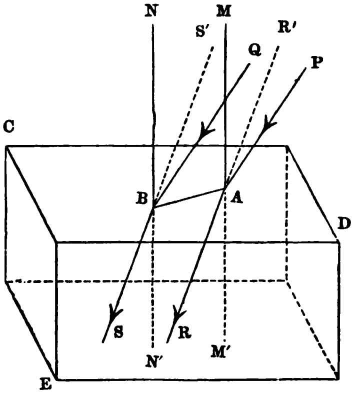
It will be seen by the figure of the last proposition that if the angle of incidence SAN had increased up to 90°, the angle of refraction RAN' would have increased to a less angle than 90°, and the course of the ray would have been the same whichever way it had passed; but if the angle RAN' of a ray moving in the medium were increased beyond this value, the law of refraction could no longer hold good, for $\sin SAN$ can never exceed unity, and therefore the limit to $\mu \sin RAN'$ is unity, or the critical angle = angle whose sine is $\frac{1}{\mu}$.

All rays within the medium falling on the surface at a greater angle than this, undergo *total reflexion*, none of them being capable of emergence.

The change from partial to total reflexion is easily seen on the internal faces of a prism of glass. An eye beneath the surface of still water sees all objects without, crowded within a circle at the surface; of which the radius subtends at the eye the angle whose sine is $\frac{1}{\mu}\left(=\frac{3}{4}\right)$; or an angle of 48° 35′. The part of the surface outside this circle reflects brilliantly the light from objects within the water.

ART. 25. PROP. *To shew that when a pencil of parallel rays is incident on the plane surface of a refracting medium, the refracted pencil consists of parallel rays.*

Let *PA, QB* be two rays of the pencil which are incident on the plane surface *CD* of the refracting medium *CDE*, at *A* and *B* respectively. Let *AR, BS* be the refracted rays, which we have to shew to be parallel. Let *MAM′, NBN′* be the normals to the surface at *A* and *B* respectively, which are parallels; by Euclid, book xi. prop. 6.



Then ∠ of incidence PAM = ∠ of incidence QBN
and ∠ of refraction RAM' = ∠ of refraction SBN'
(by the law of refraction.)
Also, plane *PAM* is parallel to plane *QBN*
(by Euclid, book xi. prop. 15)
and the refracted rays are in the same planes respectively.

Produce the direction of *RA* to *R′*; join *A, B*, and let the plane *RAB* cut the plane *QBS* in the line *BS′*, then the line *R′AR* is parallel to *BS′* (by Euclid, book xi. prop. 16)

and $\angle S'BN = \angle R'AM$ . . (by Euclid, book xi. prop. 10)
$= \angle$of refraction RAM'
$= \angle$of refraction SBN'

$\therefore$ $S'B$ and BS are in one straight line, and BS is parallel to AR, or the refracted rays are parallel.

The same can be shewn to hold for all the other refracted rays, or the refracted pencil consists of parallel rays.

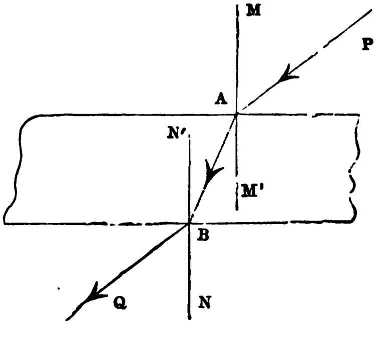
Cor. The transverse section of the pencil in the dense medium is greater than that of the incident pencil in the ratio $\frac{\cos i'}{\cos i}$. For if A be the area of the common section of the refracting surface by the pencils, we have as in Cor. Art. 3,

$$\frac{\perp \text{ area of refracted pencil}}{\perp \text{ area of incident pencil}} = \frac{A \cdot \cos i'}{A \cdot \cos i} = \frac{\cos i'}{\cos i}.$$

The intensity of the light in the refracted pencil $= n \times \frac{\cos i}{\cos i'} \times$ intensity in the incident pencil; if $n:1$ is the proportion of the transmitted to the incident rays, allowing for the loss by reflexion at the surface, and the dispersion and absorption by the dense medium.

Art. 26. Prop. *When a ray of light passes through a plate of a medium of which the refracting surfaces are plane and parallel, the direction of the emergent ray is parallel to that of the incident ray.*

Let $PABQ$ in the plane of the figure be the path of the ray, which is incident upon the first surface of the plate at A, refracted in AB and emergent from the second surface at B. Let MAM', NBN' be the normals at A and B respectively, which are parallel, since the surfaces of the plate are so.



Then BAM' the angle of refraction at $A = ABN'$ the angle of incidence on the second surface at B; and if μ be the refractive index, we have

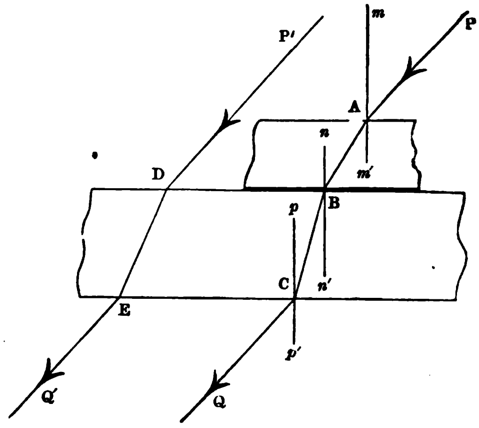
$$\begin{aligned}\sin QBN &= \mu \sin ABN' \\ &= \mu \sin BAM' \\ &= \sin PAM\end{aligned}$$

$\therefore \angle QBN = \angle PAM$, and the incident and emergent rays are parallel.

Cor. When a pencil of parallel rays traverses a plate of a refracting medium, the emergent pencil consists of parallel rays, and is of the same transverse section as the incident pencil. The intensity of the light in the emergent pencil is less than that in the incident pencil, from the dispersion and absorption by the medium, and the reflexions at the two surfaces.

Art. 27. Prop. *Having given the absolute refractive indices for two media, to find their relative refractive index.*

The relative refractive index for two given media being constant, if we find an expression for it from any incidences, it will hold for all others.



It is found by experiment that if a ray of light passes through a series of plates of dense media, all the refracting surfaces being parallel planes, that the emergent ray is parallel to the incident ray.

Let PA be a ray incident upon the first dense medium at A, which being refracted in AB enters the second medium at B, and being refracted again in BC, finally emerges in CQ, which is parallel to PA.

Let mAm', nBn', pCp', be the normals at A, B, C respectively. Then if μ_1 be the absolute refractive index of the first medium, μ_2 that of the second, and μ' the relative refractive index of the two media, we have

$$\mu_1 = \frac{\sin PAm}{\sin BAm'} \qquad \mu_2 = \frac{\sin QCp'}{\sin BCp}$$

and

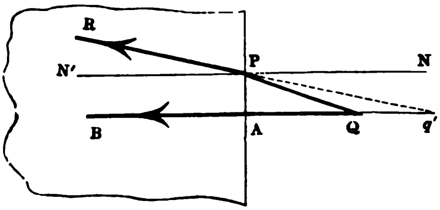
$$\begin{aligned}\mu' &= \frac{\sin ABn}{\sin CBn'} \\ &= \frac{\sin BAm'}{\sin BCp} = \frac{\sin BAm'}{\sin PAm} \times \frac{\sin QCp'}{\sin BCp} \\ &= \frac{\mu_2}{\mu_1}\end{aligned}$$

If $P'D$ be a ray parallel to PA, which enters the second medium directly at D, it will emerge from it at E in a direction EQ', which is parallel to $P'D$ and PA, and therefore to CQ; and hence the direction ED and CB in the medium must be parallel. The same holds good for any number of plates superposed on each other; or *the direction of a ray in passing through any one of the plates is parallel to the course it would have taken if it had entered the plate directly.*

We see that a small pencil of rays coming from a distant object will have a small lateral displacement on passing obliquely through a series of refracting plates, but no deviation; and since the lateral displacement is very small compared with the distance, the object appears in the same place whether the plates be interposed or not, provided the plane surfaces be accurately parallel. This is a point of consequence in the theory of Hadley's sextant.

The refraction of the earth's atmosphere causes the heavenly bodies to appear higher than they would do, if it did not exist; and when they are not very far from the zenith, we may suppose the strata of the atmosphere through which their light passes to be parallel planes, which continually decrease in density as we ascend; but the deviation of a pencil is the same as if it had entered the more dense stratum at the earth's surface at once. This deviation in astronomy is called *the refraction* of the heavenly body, and is found by observations to be in accordance with the optical rule.

ART. 28. PROP. *To find the form of a small pencil of light, after refraction directly at a plane surface of a medium.*



First. Let Q be the focus of the incident rays, QAB the ray which falls perpendicularly on the plane refracting surface at A, and therefore enters the medium without deviation. Let QP be any other ray incident at P; let NPN' be the normal to the surface at P and parallel to QAB; and let PR be the refracted ray, which being produced backwards cuts the line AQ at q'.

Then, angle of incidence $QPN = \angle PQA$
and angle of refraction $RPN' = \angle Pq'A$
also

$$\mu = \frac{\sin QPN}{\sin RPN'} = \frac{\sin PQA}{\sin Pq'A}$$

$$= \frac{Pq'}{PQ}$$

or, since the pencil is small and $PQ = AQ$, $Pq' = Aq'$ nearly,

$$\therefore Aq' = \mu \,.\, AQ$$

If we put $AQ = u$, $Aq' = u'$, we have

$$u' = \mu u$$

which gives the point q', from which the pencil diverges after refraction.

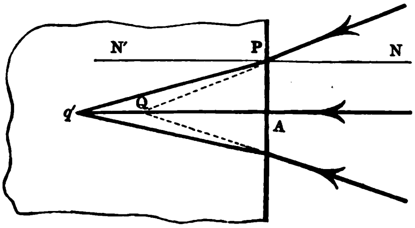
When P is not so near to A that we can put $QP = QA$, let $i = \angle$ of incidence, $i' = \angle$ of refraction; we have

$$QP = \frac{u}{\cos i} \qquad q'P = \frac{u'}{\cos i'}; \text{ and since } q'P = \mu \,.\, QP$$

$$\therefore u' = \mu \frac{\cos i'}{\cos i} u$$

which is greater than the first value of u'; or, there is aberration in the refraction of a pencil at a plane surface.

Secondly. Let Q be the focus in the figure to which the pencil converges before incidence, q' the focus to which the refracted pencil converges; then solving the triangle QPq', we find the same results as in the last case; or,

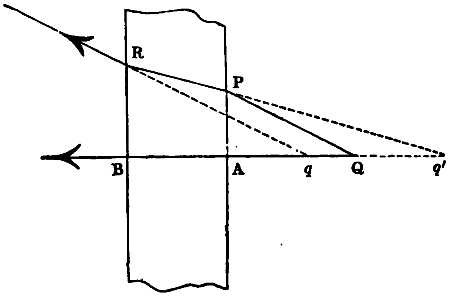


$$Aq' = \mu \, . \, AQ$$

If q' had been the origin of a pencil of diverging rays within the medium, then the emergent rays would have diverged as if they proceeded from Q.

ART. 29. PROP. *To find the form of the emergent pencil when a small direct pencil of rays passes through a plate of a refracting medium, with parallel plane surfaces.*

Let Q be the focus of the incident pencil, of which QAB the axis is incident perpendicularly on the first surface of the plate at A, and emergent perpendicularly at the second surface at B.



Let QP be another ray of the pencil incident at P, refracted in $q'PR$, and emergent at R, in the direction qR, so that q is the virtual focus of the emergent rays.

Let $QA = u$, $q'A = u'$, $Bq = v$, and $AB =$ thickness of the plate $= t$.

Then by the last proposition $\quad u' = \mu u$

and if the pencil had come the contrary way converging to q, we should have

$$Bq' = \mu \,.\, Bq$$

or $$t + u' = \mu v = t + \mu u$$

$$\therefore \qquad v = u + \frac{t}{\mu}$$

which gives the position of q.

Also $$Qq = BQ - Bq = u + t - v$$

$$= t\left(1 - \frac{1}{\mu}\right) = \frac{\mu - 1}{\mu}\, t.$$

If the plate be of glass, for which $\mu = \frac{3}{2}$, we have

$$Qq = \frac{\frac{3}{2} - 1}{\frac{3}{2}}\, t = \frac{t}{3}$$

or, the rays diverge after emergence, from a point which is nearer to the plate by $\frac{1}{3}$ the thickness.

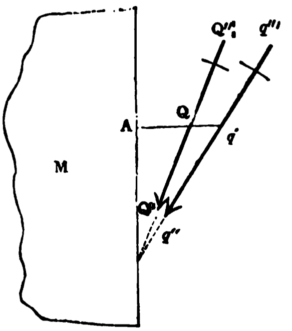
If the rays converge before incidence to a focus at q, the same proof shews us that after emergence they will converge to a focus as Q, which is distant $\frac{\mu - 1}{\mu} t$ beyond q, from the plate.

When the pencil is large, it is shown in PART II. page 60, that there is an aberration which depends on the thickness; and that when the pencil is oblique, there is confusion.

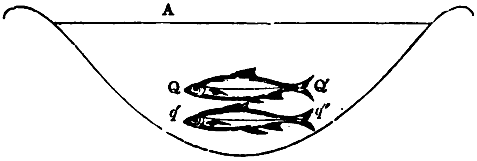
In the formation of images by pencils refracted at plane surfaces, if the incident pencils arise from an object either within or without the medium, the image will be always virtual; but if converging pencils would have formed a real image, provided the refracting medium had not interposed, then a real image will be formed by the refracted rays.

ART. 30. PROP. *To find the position of the image formed by small direct pencils from an object, which have been refracted at the plane surface of a dense medium.*

First. Let the object be exterior to the medium M, as $Q''QQ'$ in the figure; from any point Q draw a perpendicular line to the surface AQq', and make $Aq' = \mu . AQ$, then q' is the point from which the refracted rays diverge, and the virtual image of Q: the same being performed for all the other points in the object, we have an image $q'''q'q''$, as in the figure, more distant from the surface than the object; and their directions produced meet at the surface.



Secondly. Let the object be within the medium, as, for instance, in water, as in the figure. If the rays diverged from an object, as $q'q''$, parallel to the surface, by drawing perpendiculars to the surface, and measuring the distances, as in the second case of Art. 28, we should find the position of the virtual image from which the emergent rays proceed, as QQ', also parallel to the surface, and nearer to it than the object by the distance



$$Aq' - AQ$$

$$= Aq'\left(\frac{\mu - 1}{\mu}\right)$$

Therefore a small object in water appears to an eye over the surface to be nearer to it by a quantity

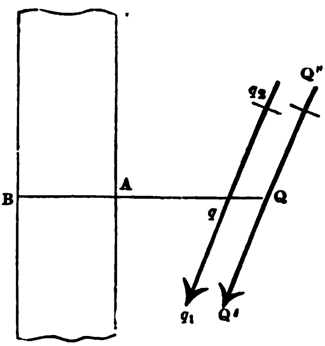
$$Aq'\left(\frac{\frac{4}{3} - 1}{\frac{4}{3}}\right) = \frac{Aq'}{4}$$

than it really is.

If pencils of rays had converged to form a real image at QQ', if the medium had not interposed, they would form a real image at $q'q''$.

If the object had been inclined to the surface, the image would also have been inclined, but at a less angle.

Art. 31. Prop. *To find the position of the image of an object when the pencils have been refracted by a plate with parallel plane surfaces.*



Let $Q'QQ''$ be the object, and drawing a perpendicular $QqAB$ to the surfaces of the medium from any point Q, make $Qq = AB\left(\frac{\mu-1}{\mu}\right)$ as found in Art. 29, we have q the focus of the rays emerging from the plate. The same being done for all other points in the object, we have an image equal and similar to it, as $q_1 q q_2$ in the figure.

In this case the only effect has been to make the object appear, to an eye on the opposite side of the plate, to be drawn nearer to it by the distance $AB\left(\frac{\mu-1}{\mu}\right)$.

When the incident pencil is not very small, there is aberration, which varies directly as the thickness of the plate and the square of the half diameter of the pencil; so that for a very thin plate it is insensible, however large the pencil may be.

When the pencil is oblique, the confusion depends on the obliquity, and is also directly as the thickness of the plate; so that in thin plates it is insensible.

An *optical prism* differs from a geometrical one in two of the inclined faces only being requisite to produce the optical effects. The ordinary form used is a triangular prism of glass, and in the very fine ones two of the faces only are polished.

Those made at the manufactory of Utzschneider and Fraunhofer at Munich* are pre-eminent for the purity and homogeneity of the glass, as well as the accurate planeness of the surfaces. The use of prisms of this quality will be seen from the articles on the chromatic dispersion.

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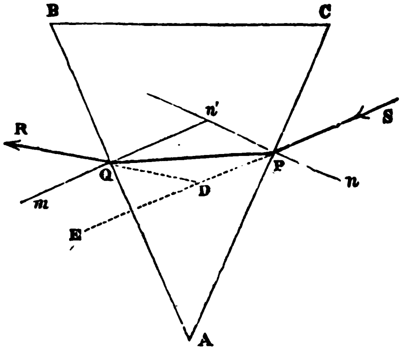
* Now carried on by Messrs. Merz and Mahler.

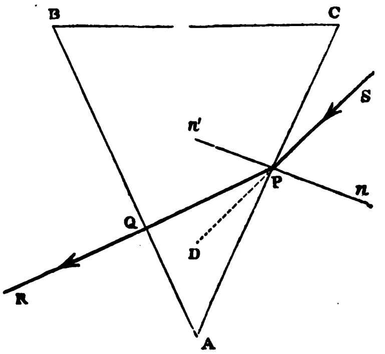
The common prisms which are sold in the opticians' shops are generally sufficient for the most ordinary experiments only; better prisms may be made with two rectangular pieces of plate glass firmly set to form two sides of a triangular box, which can be filled with water, spirit of turpentine, or other liquid.

The *angle of a prism* is the angle between the plane refracting surfaces; and *the edge* of the prism is the line in which they meet.

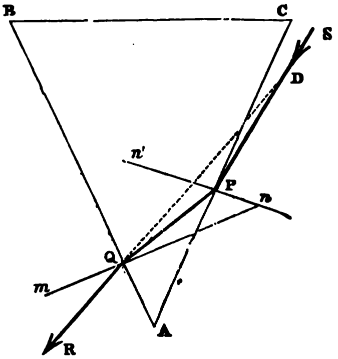
Art. 32. Prop. *When a ray of light passes in a perpendicular section through a prism of a dense refracting substance, the deviation is always towards the thicker part of the prism.*



Let *ABC* in the figures represent the perpendicular section of the prism by the plane of the paper, and let *BAC* be the refracting angle of the prism. Let *SP* be the incident ray in the figures, *PQ* the refracted ray within the prism, *QR* the emergent ray.



The angle of incidence at *P* may be such that the refracted ray *PQ* may either make an acute angle with *AQ* at incidence on the second surface at *Q*, as in the first figure; or fall perpendicularly on the second surface, as in the second figure; or, lastly, may make an obtuse angle with *AQ*, as in the third figure.



Let nn' be the normal to the surface at P, mQ that at Q, meeting nPn' at n' in the first figure, and at n in the third.

In the first figure let the directions of the incident and emergent rays meet at D, making $\angle QDE$ the deviation. The refractions at P and Q are both towards the thicker part of the prism, and therefore the whole deviation, which is the sum of the bendings at P and Q, is so also.

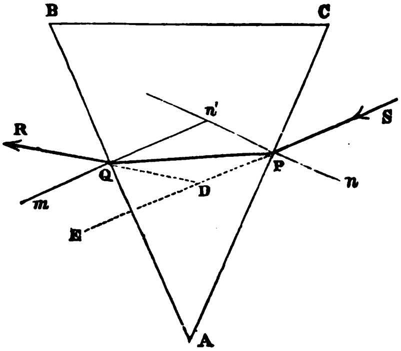
In the second figure, the incidence on the second surface being perpendicular, the ray emerges perpendicularly, and the $\angle QPD$ between the directions of the incident and emergent rays is the deviation, and towards the thicker part of the prism.

In the third figure let D be the point where the directions of the incident and emergent rays meet; the refraction at P is towards the thicker part, and at Q it is from it; but the angle of refraction, $n'PQ$ at P, is greater than the angle of incidence on the second surface nQP, since it is the exterior angle of the triangle PnQ, and to the greater angle belongs the greater deviation; therefore the deviation at P towards the thicker part is greater than the deviation at Q towards the thinner part, or the whole deviation QDP is towards the thicker part.

If we had taken the rays coming in the contrary direction, or had taken RQ for the incident ray, varying in the directions of incidences, we should have had the same results.

Cor. If the prism had been *rarer* than the surrounding medium, the refractions would have been of an opposite kind, and the deviation towards the *thinner* part of the prism.

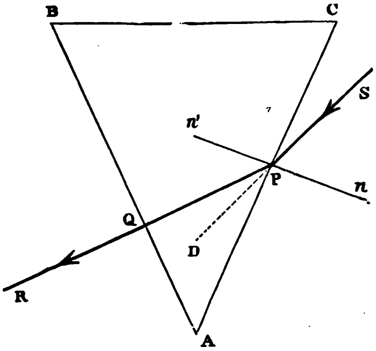
Art. 33. Prop. *To show that the algebraic sum of the angles of refraction at the first surface and of incidence at the second* *surface of a prism equals the refracting angle of the prism; and to find an expression for the deviation.*



Using the same figures as in the last proposition, we have in the annexed figure the quadrilateral $PAQn'$, right-angled at P and Q; and therefore

$$\begin{aligned} \angle PAQ + \angle Pn'Q &= 2 \text{ right angles} \\ &= \angle Pn'Q + \angle n'PQ + \angle n'QP \\ \therefore \angle n'PQ + \angle n'QP &= \angle PAQ; \end{aligned}$$

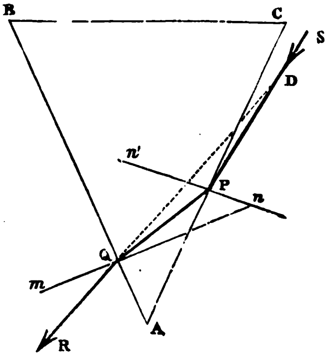
or the sum of the angles of refraction at the first surface and incidence on the second surface equals the angle of the prism.



In the annexed figure, the angle of incidence on the second surface = 0, and

$$\begin{aligned} \angle n'PQ + \angle APQ &= \text{a right angle} \\ &= \angle PAQ + \angle APQ \\ \therefore \angle n'PQ &= \angle PAQ; \end{aligned}$$

and the proposition holds good.



In the annexed figure, the angle mnn' between the normals equals the angle between the surfaces, or,

$$\angle mnn' = \angle PAQ$$

and
$$\begin{aligned}\angle n'PQ &= \angle n'nQ + \angle PQn\\ &= \angle PAQ + \angle PQn\end{aligned}$$

or,
$$\angle n'PQ - \angle PQn = \angle PAQ.$$

The angle PQn being here measured on the other side of the normal to the first case, becomes negative, and the proposition holds good as before.

Let d be the deviation, α the angle of the prism, i the angle of incidence at the first surface, i' the angle of refraction, e' the angle of incidence at the second surface, e the angle of emergence.

Then, in the first figure,
$$\begin{aligned}d &= \angle QDE\\ &= \angle QPD + \angle PQD\\ &= (i - i') + (e - e')\\ &= i + e - (i' + e')\\ &= i + e - \alpha.\end{aligned}$$

In the second figure,
$$\begin{aligned}d &= \angle QPD\\ &= i - i'\\ &= i - \alpha\end{aligned}$$

which might be derived from the last case by making $e = 0$.

In the third figure,
$$\begin{aligned}d &= \angle QDP\\ &= i - \angle \text{ between } nn' \text{ and } QD\\ &= i - (\angle PQD + \angle n'PQ)\\ &= i - (e - e' + i')\\ &= i - e - (i' - e')\\ &= i - e - \alpha\end{aligned}$$

which might have been derived from the first formula by taking e negative as falling on the other side of the normal.

ART. 34. PROP. *To find expressions for the deviation in terms of the angle of the prism, and the refractive index.*

There are three cases in which the deviation is expressed very simply: first, when the angle of the prism is small, and the ray passes with very little obliquity through it, or the angles of incidence and emergence are small; secondly, when the ray is incident or emergent perpendicularly; and, thirdly, when the angles of incidence and emergence are equal.

Taking as the general formula

$$d = i + e - \alpha$$

we have also $\sin i = \mu \sin i', \qquad \sin e = \mu \sin e'$

and $\alpha = i' + e'.$

When i and e are small, as well as α, we have, since the arcs may be put for their sines,

$$i = \mu i', \qquad e = \mu e' \text{ nearly}$$

and
$$\begin{aligned} \therefore d &= \mu (i' + e') - \alpha \\ &= (\mu - 1)\,\alpha \end{aligned}$$

or the deviation is in this case constant very nearly; and when $\mu = \frac{3}{2}$ for glass, is half the angle of the prism.

When the ray emerges perpendicularly, we have

$$e = 0, \qquad e' = 0$$

$$\therefore \alpha = i', \quad \text{and } \sin i = \mu \sin i' = \mu \sin \alpha$$

whence
$$\begin{aligned} i &= \text{arc}\,(\text{sine} = \mu \sin \alpha) \\ &= \sin^{-1} (\mu \sin \alpha) \\ \therefore d &= \text{arc}\,(\text{sine} = \mu \sin \alpha) - \alpha \\ &= \sin^{-1} (\mu \sin \alpha) - \alpha. \end{aligned}$$

When the angles of incidence and emergence are equal, we have $i = e, \qquad i' = e', \qquad \text{and } \alpha = 2i'$

and $d = 2i - \alpha$

also
$$\begin{aligned} \sin i &= \mu \sin i' = \mu \sin \frac{\alpha}{2} \\ i &= \text{arc}\left(\text{sine} = \mu \sin \frac{\alpha}{2}\right) \\ &= \sin^{-1}\left(\mu \sin \frac{\alpha}{2}\right) \\ \therefore d &= 2\,\text{arc}\left(\text{sine} = \mu \sin \frac{\alpha}{2}\right) - \alpha \\ &= 2 \sin^{-1}\left(\mu \sin \frac{\alpha}{2}\right) - \alpha. \end{aligned}$$

ART. 35. PROP. *To investigate expressions for determining the value of the refractive index for a ray of light passing through a prism, of which the angle and the deviation in one of the three simple cases are known.*

When the angle of the prism and the angles of incidence and emergence are small, the formula

$$d=(\mu-1)\alpha$$

gives us

$$\mu=\frac{d+\alpha}{\alpha}$$

and hence μ is known when d and α are known.

When the ray emerges perpendicularly, we have the formulæ

$$d=i-\alpha, \quad \text{and} \quad i'=\alpha$$

$$\therefore \mu=\frac{\sin i}{\sin i'}=\frac{\sin(d+\alpha)}{\sin\alpha}$$

which gives μ when d and α are given.

When the angles of incidence and emergence are equal, we have the formulæ

$$d=2i-\alpha, \quad \text{and} \quad i'=\frac{\alpha}{2}$$

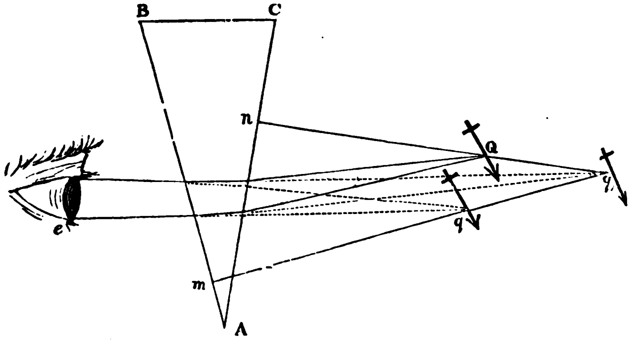
$$\therefore \mu=\frac{\sin i}{\sin i'}=\frac{\sin\left(\frac{d+\alpha}{2}\right)}{\sin\frac{\alpha}{2}}$$

This last formula is the one which is to be used when μ is required to be determined with great accuracy from experimental measures of the angle of the prism and the deviation of a ray of light of a given colour. The deviation in this case is a minimum, as is shown in PART II. page 62; and the confusion is very small when the ray passes through but little thickness, as near the edge of the prism.

Though several other methods of determining μ experimentally are applicable with advantage in particular circumstances, yet they are very inferior in accuracy to this method, when a prism of the medium can be used as above described.

Art. 36. Prop. *To find the form and position of the image of an object placed before a thin prism of a small refracting angle.*

We suppose the rays by which the image is formed to be incident and emergent so nearly perpendicularly to the surfaces that we may use the formulæ for small direct pencils in finding the position of corresponding foci.



Let BAC be the prism of small refracting angle; and let an arrow at Q, as in the figure, be the object. Draw $nQq_{,}$ a normal to the first surface through any point Q, and take

$$nq_{,} = \mu . nQ$$

then after refraction the rays in the prism will diverge as if from $q_{,}$.

Draw $q_{,}qm$ a normal to the second surface of the prism through $q_{,}$, and take

$$mq = \frac{mq_{,}}{\mu}$$

then after emergence the rays will diverge as if they had come from q; and the same being performed for the other points in the object, we find the corresponding points in the virtual image at q, as in the figure. Neglecting the thickness of the prism, we may consider the image as appearing to an eye at e in a different direction, but nearly at the same distance and of the same form as the object.

What is here shown for a thin prism holds good for a prism of any angle when the image is formed by small pencils passing through the prism at the angle of *minimum deviation,* as will be shown when we consider the *oblique* refraction of pencils.

We have to remember that we consider hitherto the light subject to refraction to be homogeneous, or of one colour only. Such light it is very far from easy to procure; the purest artificial light being obtained from the spirit-of-wine lamp, by putting a few grains of common salt on the wick, when the flame becomes of a very homogeneous orange light, though unfortunately not of great intensity. If the arrow in the figure were illuminated with this light, we should see a distinct image, as found above. The reason of the image appearing coloured and indistinct when the object is illuminated with ordinary light will be understood when the chapter on chromatics has been studied.

EXAMPLES.

Ex. 1. Show that a ray of light entering a medium for which the refractive index is 1·4142 ($=\sqrt{2}$), cannot have a greater deviation than 45°.

Ex. 2. On looking into a plate of glass ($\mu=\frac{3}{2}$), the thickness appears to be $\frac{2}{3}$ of an inch; what is the real thickness?

Ex. 3. A straight rod being partly immersed in a fluid, the part out of the fluid making an angle α with the surface, the image of the part immersed makes an angle α', such that

$$\tan \alpha' = \frac{\tan. \alpha}{\mu}$$

Ex. 4. A piece of glass being placed in water, show that the refractive index for a ray passing from the water into the glass is $\frac{9}{8}$, and from the glass into the water is $\frac{8}{9}$.

Ex. 5. Show that when a prism of glass of a small refracting angle is immersed in water, the deviation of a ray passing through it is only one-fourth of what it is in air.

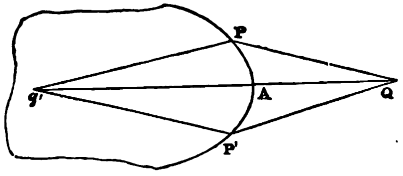
Ex. 6. Show that if the angle of a prism of any substance be greater than twice the critical angle, no ray of light can pass through the prism. Explain from this why we never see through two contiguous faces of a cube of glass.

Ex. 7. The angle of a prism of glass being 60°, show that the minimum deviation of a ray of light passing through it is $37^\circ\ 10'$ nearly; having given that $\sin(48^\circ\ 35') = {\cdot}75$ nearly.

CHAPTER V.

ON THE REFRACTION OF HOMOGENEOUS LIGHT AT CURVED SURFACES OF MEDIA.

The surface of accurate refraction, or that which refracts to a point q' a pencil of rays diverging from a point Q, in the figure, is formed (see Part II. page 68) by the revolution of a transcendental curve about the axis QAq', which is such that for any incident ray QP, we have

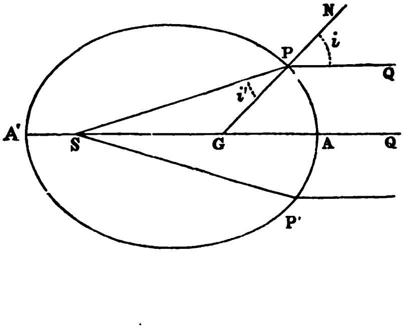


$$QP + \mu \,.\, q'P = QA + \mu \,.\, q'A$$
$$= \text{a constant quantity for every ray.}$$

In a few particular cases the curve becomes one of the conic sections; but the surfaces of accurate refraction are of less importance than those of accurate reflexion, because there are means of correcting the errors in *dioptrical* instruments, by the errors of one refraction compensating for those of another, which do not exist in the same way in *catoptrical ones.*

Art. 37. Prop. *If a pencil of parallel rays fall parallel to the axis upon a convex surface of a refracting medium which is a prolate spheroid, the eccentricity of the generating ellipse being the reciprocal of the refractive index, or* $e = \frac{1}{\mu}$, *then the pencil will be refracted accurately to the further focus.*

Let QA be the ray incident perpendicularly at A, and which passes along $AGSA'$ the axis of the spheroid; QP another ray of the pencil incident at P, and refracted in PS. Let PG be the normal to the ellipse at P.



Then by the property of the ellipse, if S be the further focus, we have $SG=e\,.\,SP$.

Produce the direction of PG to N; then

$$\angle \text{ of incidence } QPN=\angle AGP=i, \text{ say};$$

and if i' be the angle of refraction, we have

$$\frac{\sin i}{\sin i'}=\mu=\frac{1}{e}$$

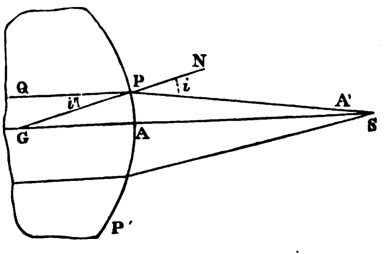
$$=\frac{SP}{SG}$$

$$=\frac{\sin AGP}{\sin SPG}$$

$$\text{and} \quad i=\angle AGP \qquad \therefore i'=\angle SPG$$

and SP is the refracted ray corresponding to QP; similarly every other refracted ray will pass through S. Reciprocally, if a pencil of rays diverged within the medium from S, they would after emergence be parallel.

ART. 38. PROP. *If a pencil of rays diverging from a point which is the further focus of an hyperboloid of revolution, about the major axis, which forms a convex refracting surface of a dense medium, the eccentricity of the generating hyperbola being equal to the refractive index or $e=\mu$, then the rays within the medium will be parallel to each other.*



Let S be the point from which the rays diverge, and $SA'AG$ the one passing in the axis of the hyperboloid, without changing its direction on entering the dense medium at A. Let SP be any other ray incident on the hyperboloid at P, and PG the normal, which is produced to N.

Then from the property of the hyperbola $SG=e\,.\,SP$; and as in the last proposition

$$\frac{\sin i}{\sin i'}=\mu=e$$

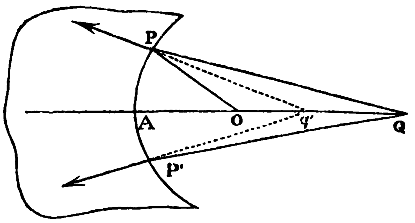
$$=\frac{SG}{SP}$$

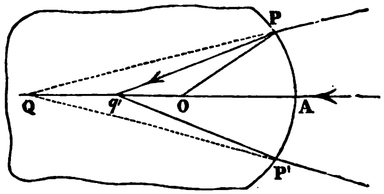
$$=\frac{\sin SPG}{\sin SGP}$$

and $\angle SPN=i \;\therefore \angle SGP=i'$; and if we draw PQ parallel to AG, it will be the direction of the ray within the medium; similarly every other ray within the medium will be parallel to AG.

Reciprocally, a pencil of rays within the medium, parallel to the axis of the hyperboloid, would after emergence converge to S.

ART. 39. PROP. *The refraction at a spherical surface is accurate, for a diverging pencil at a concave surface and for a converging pencil at a convex surface, when the radius is a mean proportional between the distances of the conjugate foci from the center.*





Let Q be the point from which the pencil diverges when it falls on the concave spherical surface in the upper figure, and the point towards which it converges when incident on the convex surface of the medium in the lower figure.

If the radius OA and distance Oq' be in the relation to OQ as follows:

$$QO:AO::\mu:1$$

$$::AO:q'O$$

which gives $QA=\mu . q'A$, then q' is the accurate focus of the refracted pencil.

For the triangles QPO, $q'PO$, having a common angle at O, and the adjacent sides proportionals, are similar, by Euclid, book vi. prop. 6; for $AO = OP$,

and $$\frac{QO}{QP} = \frac{OP}{q'P}$$

but $$\frac{q'O}{OP} = \frac{1}{\mu}$$

$$\therefore \frac{QO}{QP} = \frac{OP}{q'P} \times \mu \frac{q'O}{OP} = \mu \frac{q'O}{q'P}$$

Now in triangle QPO

$$\frac{QO}{QP} = \frac{\sin QPO}{\sin POA}$$

and in triangle $q'PO$

$$\frac{q'O}{q'P} = \frac{\sin q'PO}{\sin POA}$$

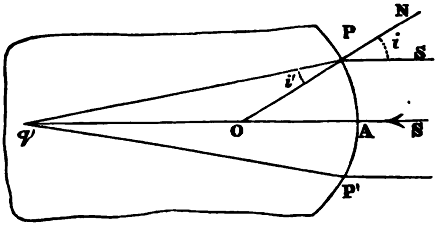
$$\therefore \sin QPO = \mu \,.\, \sin q'PO$$

and QPO is the angle of incidence for any ray; and therefore $q'PO$ is the angle of refraction, and q' is the accurate focus of the refracted pencil.

Cor. The reciprocal of the above evidently holds good, that a pencil within the medium having q' for focus will after emergence have Q for accurate focus.

When a pencil of rays is refracted at a spherical surface of a medium, the refraction is not accurate, except in the cases of the last proposition, and when the pencil converges towards or diverges from the center; there being generally aberration, as in mirrors, and a caustic similarly formed; but at the focus, which is the cusp of the caustic, the refracted rays are collected approximately to a point for a pencil of *small* magnitude; and as spherical and plane surfaces only are used in dioptrical instruments, it is necessary to know the properties of such surfaces: the corrections will be found treated in Part II.

Art. 40. Prop. *To find the focus of the refracted rays* *when a small pencil of parallel rays is incident directly on a convex spherical surface of a dense medium.*



Let PAP' be the convex surface of the dense medium, of which O is the center. Let SAO be the ray of the pencil which passes through O, and therefore is incident perpendicularly at A, and enters the medium without deviation. Let SP be another of the parallel rays refracted in Pq', and meeting the ray SAO in q', which is the focus of the refracted rays required.

Draw the radius OP, and produce it to N.
Then angle of incidence $SPN=\angle AOP=i$,
and angle of refraction $=\angle OPq'=i'$.
Let μ be the refractive index, and r be the radius OA or OP. In the triangle $q'PO$ we have

$$\therefore \frac{\sin AOP}{\sin q'PO}=\frac{\sin i}{\sin i'}=\mu=\frac{q'P}{q'O}$$

When the pencil is small, we have $q'P=q'A$ very nearly, which is called the *principal* focal length of the refracting surface. Let $q'A=f'$, then

$$\frac{q'A}{q'O}=\mu=\frac{f'}{f'-r}$$

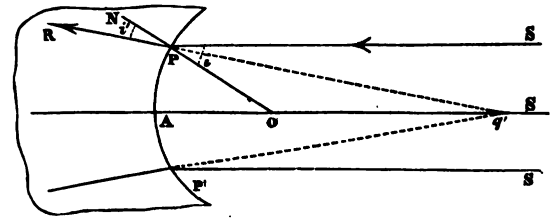
and

$$f'=\frac{\mu r}{\mu-1}$$

If the refracting medium be glass and $\mu\frac{3}{2}=$, then $f'=3r$.

ART. 41. PROP. *To find the focus of the refracted rays when a small pencil of parallel rays is incident directly on a concave spherical surface of a dense medium.*

Let PAP' be the concave refracting surface, of which O is the center. Let SOA be the ray which enters the medium with-



out deviation; *SP* a ray incident at *P*, and refracted in *PR*, which being produced meets the line *SOA* in q'. The pencil being small, the distance Aq' is the principal focal length f', which is measured in the contrary direction to that in the former case; the focus now being virtual.

Draw the radius *OP*, and produce it to *N*.

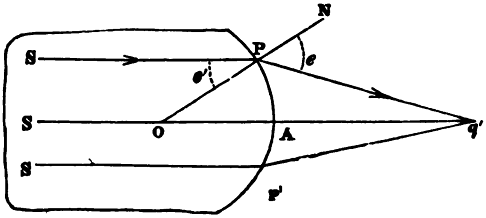
The angle of incidence $SPO = \angle AOP = i$

angle of refraction $NPR = \angle OPq' = i'$.

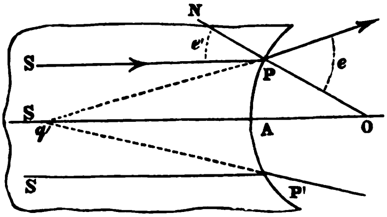
Putting $AO = r$, and solving the triangle $q'PO$ as in the last proposition, we have the same result; or,

$$Aq' = f' = \frac{\mu r}{\mu - 1}$$

ART. 42. PROP. *To find the focus of the emergent rays when a small pencil of parallel rays within a dense medium is incident upon a spherical surface.*



Let the surface of the medium be convex in the upper and concave in the lower figure, and *SA*, *SP* rays within the medium which are parallel, and *SA* passing through the center *O*.



Putting the same letters for the same things as in the previous propositions, and

$e' = \angle$ of incidence within the medium,

$e = \angle$ of emergence.

we have, as before,

$$\frac{\sin. e}{\sin. e'} = \mu = \frac{q'O}{q'P}$$

which for a small pencil becomes ultimately

$$\mu = \frac{q'O}{q'A}$$

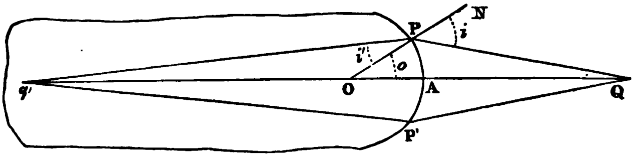
$$= \frac{f'+r}{f'}$$

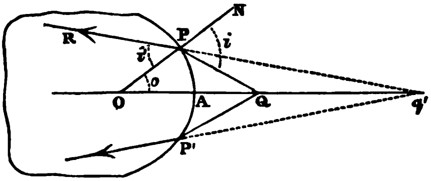
whence

$$f' = \frac{r}{\mu - 1}$$

Cor. Reciprocally, if the rays had diverged from q' in the first, or converged towards q' in the second figure, they would have been refracted parallel within the medium.

Art. 43. Prop. *To find the form of the refracted pencil when a small diverging pencil of rays is incident directly on a convex spherical surface of a dense medium.*





Let PAP' be the spherical surface of which the center is O, Q the focus of the incident rays, and there will be two cases, as in the two figures, according as AQ is greater or less than the value found for Aq' in the last proposition; q' the focus of the refracted rays, which is real in the first figure and virtual in the second. Let QAO be the ray which enters the medium without deviation at A, QP another ray of the incident pencil, which is refracted in Pq' in the first figure, and in $q'PR$ in the second.

Draw the radius OP, and produce it to N.

Then, the angle of incidence $= \angle QPN = i$

the angle of refraction $= \left\{\begin{matrix} \angle q'PO \\ \angle RPO \end{matrix}\right\} = i'$

let the angle POA $= o$

from the triangle QOP we have $\dfrac{\sin. i}{\sin. o} = \dfrac{QO}{QP}$

from the triangle $q'OP$ . . . . $\dfrac{\sin. o}{\sin. i'} = \dfrac{q'P}{q'O}$

by compounding,

$$\frac{\sin. i}{\sin. i'} = \mu$$

$$= \frac{QO}{QP} \times \frac{q'P}{q'O}$$

or,

$$\frac{QO}{QP} = \mu . \frac{q'O}{q'P}.$$

Since the pencil is very small, $QP = QA$, and $q'P = q'A$ nearly; let $QA = u$, $q'A = u'$, $AO = r$,

we have

$$\frac{QO}{QA} = \mu . \frac{q'O}{q'A}$$

or

$$\frac{u+r}{u} = \mu . \frac{u' \mp r}{u'}$$

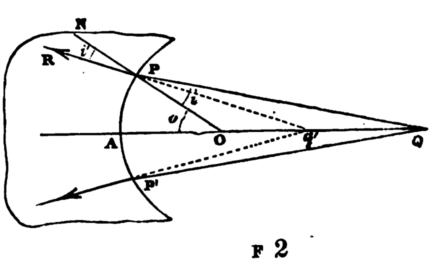
the upper sign referring to the upper figure, and the lower sign to the lower; whence, dividing by r,

$$\pm \frac{\mu}{u'} = \frac{\mu - 1}{r} - \frac{1}{u}$$

$$= \frac{\mu}{f'} - \frac{1}{u}$$

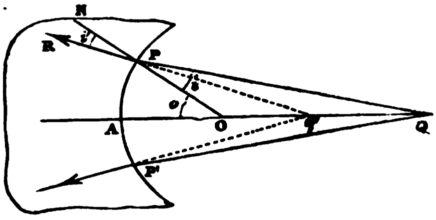
which gives q' the focus of the refracted rays.

ART. 44. PROP. *To find the form of the refracted pencil when a small diverging pencil is incident directly on a concave spherical surface of a dense medium.*



Let PAP' be the concave spherical surface of the medium, of which O is the center. Let Q be the point from which the incident rays diverge; QOA the ray passing through

O, and entering the medium at *A* without deviation; *QP* another ray incident at *P*, and refracted in *PR*, which being produced backward, meets *QA* in q', which is the virtual focus of the refracted rays.



Then the angle of incidence $= \angle QPO = i$

the angle of refraction $= \angle RPN = \angle q'PO = i'$

let the angle $POA = o$

from the triangle *QPO* we have $\dfrac{\sin. i}{\sin. o} = \dfrac{QO}{QP}$

from the triangle $q'PO$ . . . . $\dfrac{\sin. o}{\sin. i'} = \dfrac{q'P}{q'O}$

by compounding, $\dfrac{\sin. i}{\sin. i'} = \mu$

$$= \frac{QO}{QP} \times \frac{q'P}{q'O}$$

or, $$\frac{QO}{QP} = \mu . \frac{q'O}{q'P}$$

The pencil being small, we may put $QA = QP$, and $q'A = q'P$;

$$\therefore \frac{QO}{QA} = \mu \frac{q'O}{q'A}$$

putting $u = QA$, $u' = q'A$, $OA = r$, as before,

we have $$\frac{u-r}{u} = \mu . \frac{u'-r}{u'}$$

whence $$\frac{\mu}{u'} = \frac{\mu-1}{r} + \frac{1}{u}$$

$$= \frac{\mu}{f'} + \frac{1}{u}$$

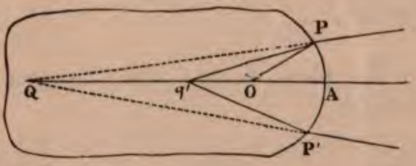
which gives q' the virtual focus from which the refracted pencil diverges.

Cor. Reciprocally, if the pencil within the medium had converged to q', it would after emergence converge to *Q*.

The cases of converging pencils incident upon convex or concave surfaces may be solved directly, as in the two last propositions; but as every case of the direct refraction of small pencils at plane or spherical surfaces may be deduced from either of these by taking lines negative, which fall in a direction opposite to what they do in the case taken for standard, we will apply that method to converging pencils, together with the other cases, and take for our standard case the one in which a pencil diverging from a distant point is incident on a convex surface; for which the formula is

$$\frac{\mu}{u'} = \frac{\mu - 1}{r} - \frac{1}{u}$$

Art. 45. *To adapt the formula to the case of a converging pencil incident on a convex surface.*



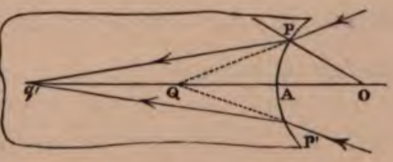
Since the pencil converges, Q falls on the opposite side of A to the standard case; and therefore u must be taken negative, and the formula is

$$\frac{\mu}{u'} = \frac{\mu - 1}{r} + \frac{1}{u}$$

If Q had fallen between A and O, we should still have had q' between Q and O; and the formula shows us that u' cannot change sign; or q' is always on the same side of A.

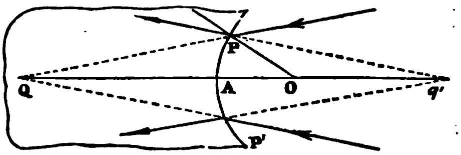
Cor. If the pencil within the medium diverged from q', it would after emergence diverge from Q.

Art. 46. *To adapt the standard formula to the case of a converging pencil incident on a concave surface.*



Since the incident pencil is converging, and the surface concave, we must change the signs of both u and r, and the formula becomes

$$\frac{\mu}{u'} = -\frac{\mu-1}{r} + \frac{1}{u}$$



We see that there will be two cases; as in the figures; for u' will be positive or negative, as $\frac{1}{u}$ is greater or less than $\frac{\mu-1}{r}$.

Cor. If the rays within the medium diverged from q' in the first, or converged towards q' in the second figure, they would after emergence diverge from Q.

Art. 47. *To adapt the standard formula to the case of a plane surface.*

The plane surface must be considered as spherical to radius infinity;

or, $r = \infty \quad \therefore \frac{\mu-1}{r} = 0$

and $\frac{\mu}{u'} = -\frac{1}{u}$

or $u' = -\mu u$

as found in Art. 28, the negative sign showing that u' is to be measured in the same direction as u, or the contrary to what it is in the standard case.

Art. 48. *To adapt the standard formula to the case of parallel rays incident on a concave surface.*

Since the rays are parallel, we must consider Q at an infinite distance, or $u = \infty$ and $\frac{1}{u} = 0$; also we have r negative, therefore the formula becomes

$$\frac{\mu}{u'} = -\frac{\mu-1}{r}$$

where the negative sign shows that u' is to be measured in the contrary direction to the standard case, which accords with the result of Art. 41.

ART. 49. *To adapt the standard formula to the case of parallel rays incident at a convex surface, and also of a pencil diverging from a near point.*

For a pencil of parallel rays, we have $u=\infty$ and $\frac{1}{u}=0$

$$\therefore \quad \frac{\mu}{u'}=\frac{\mu-1}{r}$$

and $$u'=f'=\frac{\mu r}{\mu-1}, \text{ as found in Art. 40.}$$

Again, if Q be so near A that u is less than $\frac{r}{\mu-1}$, the value found for f' in Art. 42, we have

$$\frac{\mu}{u'}=-\left(\frac{1}{u}-\frac{\mu-1}{r}\right)$$

and u' is negative, q' falling on the same side of A as Q, in accordance with the lower figure of Art. 43.

Taking u' measured in the contrary direction, or negative, we have

$$\frac{\mu}{u'}=\frac{1}{u}-\frac{\mu-1}{r}$$

ART. 50. PROP. *To trace the corresponding changes of position of the conjugate foci Q and q'; and to show that they coincide at A and O.*

Referring to the first figure of Art. 43, we see that if Q were at an indefinitely great distance, the incident rays being parallel, q' would be at the principal focus, as found in Art. 40. If Q then approaches the surface, the angle of incidence increasing, the angle of refraction increases also, and q' moves further from A. If Q comes to the distance f', found in the first figure of Art. 42, q' has moved to an infinite distance, and the rays within the medium are parallel. If Q still approaches nearer to A, q' falls on the same side of A with Q, as in the second figure of Art. 43. If Q comes to the point A upon the medium itself, the refracted rays diverge from the same point, or Q and q' coincide at A. If Q falls between A and O, q' falls between Q and O, as found in Art. 45. If Q comes to O, the incident rays being all perpendicular to the surface, the refracted rays will be so also, and q' coincides with Q at O. If Q falls on the further side of O from A, q' falls between Q and O, as shown in Art. 45. If Q be at an infinite distance, the rays become again parallel, and q' is at the principal focus.

We see that Q and q' always move in the same direction, and coincide at A and O.

The changes in the relative positions of Q and q' may be traced from the analytical discussion of the expression

$$\frac{\mu}{u'}=\frac{\mu-1}{r}-\frac{1}{u}$$

which refers to the first figure of Art. 43; thus,

if $u=\infty$ then $\frac{\mu}{u'}=\frac{\mu-1}{r}$ and $u'=f'$

or, q' is at the principal focus.

" $u>\frac{r}{\mu-1}$ .... $\frac{\mu}{u'}>0$ ... $u'<\infty$

or, q' is beyond the principal focus from A.

" $u=\frac{r}{\mu-1}$ .... $\frac{\mu}{u'}=0$ ... $u'=\infty$

or, q' is at an infinite distance.

" $u<\frac{r}{\mu-1}$ .... $\frac{\mu}{u'}$ is $-^{\text{ve}}$ ... u' is $-^{\text{ve}}$

or, q' falls on the other side of A.

" $u=0$ .... $\frac{\mu}{u'}=\infty$ ... $u'=0$

or, q' coincides with Q at A.

" $u-^{\text{ve}}$ .... $\frac{\mu}{u'}$ is $+^{\text{ve}}$ ... u' is $+^{\text{ve}}$

or, q' falls on the original side of A.

if $u=-r$ then $\frac{\mu}{u'}=\frac{\mu}{r}$ and $u'=r$

or, q' coincides with Q at O.

,, $u>-r$ .... $\frac{\mu}{u'}<\frac{\mu}{r}$ ... $u'>r$

or, q' falls on the other side of O.

,, $u=-\infty$ .... $\frac{\mu}{u'}=\frac{\mu-1}{r}$ ... $u'=f'$

or, q' falls at the principal focus.

In the same manner we may trace the relative positions of the conjugate foci for concave refracting surfaces.

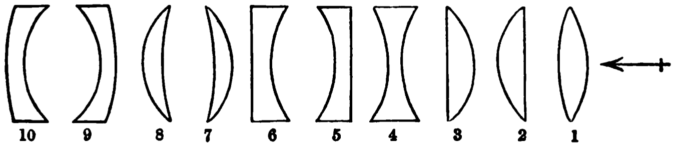
CHAPTER VI.

ON THE COMBINED REFRACTIONS OF HOMOGENEOUS LIGHT AT CURVED SURFACES, AND THE FORMATION OF IMAGES.

Definition. A *lens* is a portion of some transparent substance, as glass, crystal, &c., of which the surfaces are *generally* either both spherical, or one plane and the other spherical. Sometimes they are formed by other surfaces of revolution, which must then be specified; and if there is no mention of the particular form of the surface, we understand them to be as first mentioned.

The *axis* of a lens is the line joining the centers of the spherical surfaces when both are curved; and the line perpendicular to the plane surface which passes through the center of the other surface when one side is plane.

The following figures represent the sections of the different forms through their axes; and when the surfaces are of different kinds they are differently named according to the side on which the light first falls. We suppose the light to come in the direction of the arrow.



The form 1 is called the *double-convex* or *convexo-convex*; when the surfaces are of equal radii it is called the *equi-convex* lens.
„ 2 „ *plano-convex.*
„ 3 „ *convexo-plane.*
„ 4 „ *double-concave,* or *concavo-concave.*
„ 5 „ *plano-concave.*
„ 6 „ *concavo-plane.*

The form 7 is called the *convex meniscus.*
„ 8 „ *concave meniscus.*
„ 9 „ *convexo-concave.*
„ 10 „ *concavo-convex.*

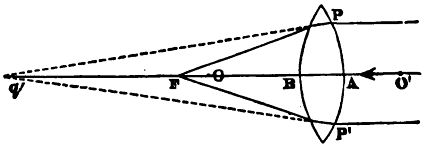
The whole series are comprised in two classes, namely, those which are *thickest* in the center, called convex lenses; and those which are *thinner* in the center than at the edge, called concave lenses.

Lenses of the same class, but of different forms, may have the same focal length; yet each has its peculiar properties in respect to aberration and coloured dispersion, which makes it more proper than the others for particular uses in optical instruments.

The thickness of a lens is generally supposed in the construction of instruments to be so small, compared with its focal length, that it may be neglected; it is, however, easy to take the thickness into account, supposing it to be given at the same time with the radii of the surfaces.

Art. 51. Prop. *To find the principal focal length of a double convex lens.*

Let $PAP'B$ be the lens, of which O is the center of the first surface; O' that of the second. Let the radius OA of the first surface $=r$; $O'B$ that of the second surface $=s$.



The incident pencil of parallel rays being small and direct, let $O'ABO$ be the ray which is the axis of the pencil passing through the lens without deviation; q' the point towards which the rays converge whilst within the lens; F the point to which they converge after emergence, or the principal focus.

First; let the thickness AB be so small as to be negligible, and

$$Aq' = Bq' \text{ nearly } = f'; \quad BF = AF \text{ nearly } = f.$$

By Art. 40, we have

$$\frac{\mu}{f'} = \frac{\mu-1}{r}$$

Supposing the rays to pass through the lens in the contrary direction, diverging from F, we have, from the second case of Art. 43,

$$-\frac{\mu}{Bq'} = \frac{\mu-1}{s} - \frac{1}{BF}$$

or

$$-\frac{\mu}{f'} = \frac{\mu-1}{s} - \frac{1}{f}$$

Adding this to the former, we have

$$0 = (\mu-1)\left\{\frac{1}{r} + \frac{1}{s}\right\} - \frac{1}{f}$$

or,

$$\frac{1}{f} = (\mu-1)\left\{\frac{1}{r} + \frac{1}{s}\right\}$$

which gives f the principal focal length required.

Again; let the thickness $AB = t$ be small, but not negligible. At the first refraction we have

$$\frac{\mu}{Aq'} = \frac{\mu-1}{r}$$

At the second refraction

$$-\frac{\mu}{Bq'} = -\frac{\mu}{Aq'-t}$$

$$= \frac{\mu-1}{s} - \frac{1}{BF}$$

but

$$\frac{\mu}{Aq'-t} = \frac{\mu}{Aq'}\left(\frac{1}{1-\frac{t}{Aq'}}\right) = \frac{\mu}{Aq'}\left(1 + \frac{t}{Aq'}\right)$$ nearly, since $\frac{t}{Aq'}$ is small.

$$= \frac{\mu}{Aq'} + \frac{\mu t}{(Aq')^2}$$

Substituting for Aq' from the first equation

$$-\frac{\mu-1}{r} - \mu t\left(\frac{\mu-1}{\mu r}\right)^2 = \frac{\mu-1}{s} - \frac{1}{BF}$$

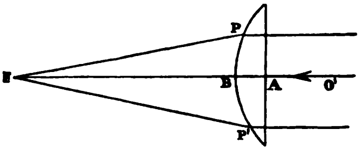
whence

$$\frac{1}{BF}=\frac{1}{f}=(\mu-1)\left\{\frac{1}{r}+\frac{1}{s}\right\}+\frac{t}{\mu}\cdot\left(\frac{\mu-1}{r}\right)^2$$

For glass and $\mu=\frac{3}{2}$, if the lens be equi-convex, and thin,

$$f=r.$$

ART. 52. PROP. *To find the principal focal length of a plano-convex lens.*

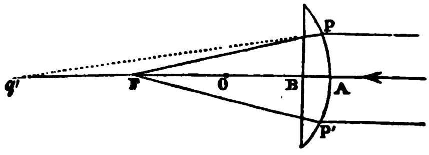


Let F be the principal focus of the lens $APBP'$, and $BF=f$. Let the radius of the second surface $=s$. All the rays of the incident pencil being perpendicular to the first surface, are bent from their original direction only at the second surface; and considering the pencil to pass the contrary way, diverging from F, we have, from the case treated in Art. 42.

$$BF=f=\frac{s}{\mu-1}$$

For glass and $\mu=\frac{3}{2}$, we have $f=2s$.

ART. 53. PROP. *To find the principal focal length of a convexo-plane lens.*



Let F be the principal focus of the convexo-plane lens $PAP'B$, of which O is the center of the first surface, and $OA=r$; let q' be the focus of the refracted rays within the lens; and let $AB=t$, $BF=f$.

At the first refraction

$$\frac{\mu}{Aq'}=\frac{\mu-1}{r}$$

At the second refraction, Arts. 47 and 28,

$$\frac{\mu}{Bq'}=\frac{1}{BF}$$

If t is so small as to be negligible, or $Bq'=Aq'$ nearly;

$$f=BF=\frac{r}{\mu-1}$$

the same value as if the lens were turned with the other side to the incident light, and plano-convex.

When t is small compared with r, but not negligible, we have

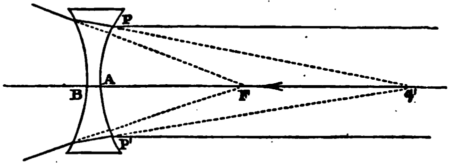
$$\frac{\mu}{Bq'}=\frac{\mu}{Aq'-t}=\frac{\mu}{Aq'}+\frac{\mu t}{(Aq')^2} \text{ as in Art. 51;}$$

and substituting for Aq'

$$\frac{1}{f}=\frac{1}{BF}=\frac{\mu-1}{r}+\frac{t}{\mu}\cdot\left(\frac{\mu-1}{r}\right)^2$$

It may appear indifferent which side of the lens be turned towards the parallel rays, since the focal length is nearly the same; but on going to second approximations, we find that the convexo-plane lens has only about one-fourth of the aberration of the plano-convex, and about two-thirds of that of an equi-convex of the same focal length. When the chromatic dispersion is not corrected, as we shall see in the next chapter, the equi-convex is generally the most desirable form of a lens.

Art. 54. Prop. *To find the principal focal length of a double concave lens.*



Let $PAP'B$ be the lens, and $q'AB$ the ray which is the axis of the pencil of parallel rays, and which passes through the lens without deviation. Let q' be the point from which the rays within the medium diverge, as in Art. 41; and F the point from which they diverge after emergence, as in Cor. Art. 46, or the principal focus. Let r be the radius of the first surface, s that of the second surface, and AB the thickness $=t$.

By Art. 41, we have at the first refraction

$$\frac{\mu}{Aq'}=\frac{\mu-1}{r}$$

By Cor. Art. 46, we have at the second refraction

$$\frac{\mu}{Bq'}=-\frac{\mu-1}{s}+\frac{1}{BF}$$

If t is so small that we may put $Aq'=Bq'$ nearly $=f'$, subtracting we have

$$0=(\mu-1)\left\{\frac{1}{r}+\frac{1}{s}\right\}-\frac{1}{BF}$$

or

$$\frac{1}{BF}=\frac{1}{f}=(\mu-1)\left\{\frac{1}{r}+\frac{1}{s}\right\}$$

If t be small compared with f, but not negligible, we have as before,

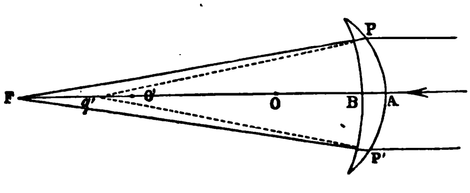
$$\frac{\mu}{Bq'}=\frac{\mu}{Aq'+t}=\frac{\mu}{Aq'}-\frac{\mu t}{(Aq')^2}$$

and substituting the value of $\frac{1}{Aq'}$, we find

$$\frac{1}{BF}=\frac{1}{f}=(\mu-1)\left\{\frac{1}{r}+\frac{1}{s}\right\}-\frac{t}{\mu}\cdot\left(\frac{\mu-1}{r}\right)^2$$

We see that the expression for the focal length is the same as for a double-convex lens, if the lens be very thin; but the focus is *real* for the convex and *virtual* for the concave lens, the former causing parallel rays to converge, and the latter causing them to diverge. We see also the analogy which lenses bear to prisms, in the deviation of a ray being always towards the thicker part of the lens.

Art. 55. Prop. *To find the principal focal length of a convex meniscus.*



Let $PAP'B$ be the meniscus, of which O and O' are the centers of the first and second surfaces respectively. Let q' be the focus of the rays within the lens, F that of the emergent rays, r and s the radii of the first and second surfaces respectively.

At the first refraction we have

$$\frac{\mu}{Aq'} = \frac{\mu-1}{r}$$

At the second refraction, by Cor. Art. 44,

$$\frac{\mu}{Bq'} = \frac{\mu-1}{s} + \frac{1}{BF}$$

Neglecting the thickness $AB=t$, we have by subtracting,

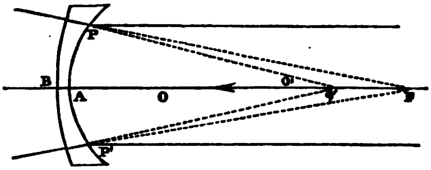
$$\frac{1}{BF} = \frac{1}{f} = (\mu-1)\left\{\frac{1}{r} - \frac{1}{s}\right\}$$

Supposing t to be small compared with f, but not negligible, we find, as in the previous propositions,

$$\frac{1}{f} = (\mu-1)\left\{\frac{1}{r} - \frac{1}{s}\right\} + \frac{t}{\mu}\cdot\left(\frac{\mu-1}{r}\right)^2$$

We see that we must have r less than s in order that the lens may be a meniscus and the focus *real*, by the analysis as well as the figure.

Art. 56. Prop. *To find the principal focal length of a concavo-convex lens.*



The lens being $PAP'B$, and the letters q', F, O, O' referring to the same points as in the previous Articles; let r and s be the radii of the first and second surfaces as before.

At the first refraction, Art. 41, we have

$$\frac{\mu}{Aq'} = \frac{\mu-1}{r}$$

At the second refraction, by Cor. Art. 45, we have

$$\frac{\mu}{Bq'} = \frac{\mu-1}{s} + \frac{1}{BF}$$

whence, neglecting the thickness,

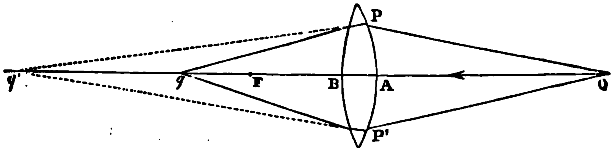
$$\frac{1}{BF}=(\mu-1)\left\{\frac{1}{r}-\frac{1}{s}\right\}=\frac{1}{f}$$

Allowing for the thickness, we find, in the same manner as before,

$$\frac{1}{f}=(\mu-1)\left\{\frac{1}{r}-\frac{1}{s}\right\}-\frac{t}{\mu}\cdot\left(\frac{\mu-1}{r}\right)^2$$

The remaining cases will be deduced as particular cases of a general formula taken as standard, but may easily be solved independently, like the preceding.

ART. 57. PROP. *To find the focus of the emergent rays when a small direct pencil of diverging rays is incident on a double-convex lens.*



Let $PAP'B$ be the lens, of which r is the radius of the first surface, s that of the second. Let Q be the focus of the incident rays, q' that of the refracted rays within the lens, and q the focus of the emergent pencil required. Let $QA=u$, $q'A=u'$, $qB=v$, $AB=t$.

At the first refraction we have, by Art. 43,

$$\frac{\mu}{u'}=\frac{\mu-1}{r}-\frac{1}{u}$$

At the second refraction, considering the rays to pass the contrary way, by the same Article,

$$-\frac{\mu}{Bq'}=\frac{\mu-1}{s}-\frac{1}{v}$$

If the thickness may be neglected, and $Bq'=Aq'=u'$ nearly, by adding the two expressions above, we find,

$$\frac{1}{v}=(\mu-1)\left\{\frac{1}{r}+\frac{1}{s}\right\}-\frac{1}{u}$$

or, $$\frac{1}{v}=\frac{1}{f}-\frac{1}{u}$$

If t be small, but not negligible, we have

$$-\frac{\mu}{Bq'}=-\frac{\mu}{u'-t}=-\frac{\mu}{u'}-\frac{\mu t}{u'^2}$$
$$=\frac{\mu-1}{s}-\frac{1}{v}$$

Adding to the first equation, we find

$$\frac{1}{v}=(\mu-1)\left\{\frac{1}{r}+\frac{1}{s}\right\}-\frac{1}{u}+\frac{\mu t}{u'^2}\cdot$$

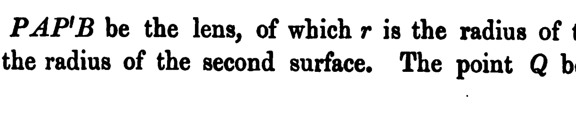
or, $$\frac{1}{v}=\frac{1}{f}-\frac{1}{u}+\frac{t}{\mu}\left(\frac{\mu-1}{r}-\frac{1}{u}\right)^2$$

We might have had Q so near to A that q' would have fallen on the same side as Q; but q' and q falling on opposite sides of the lens, the result becomes the same as above. If Q is placed so near to A that u becomes less than f, then q falls on the same side of the lens as Q, and v becoming negative, we have

$$-\frac{1}{v}=\frac{1}{f}-\frac{1}{u}.$$

In the use of convex lenses, q frequently falls in this manner on the same side of the lens as Q, being a virtual focus. The case may be deduced from the one worked out above, as the standard case for lenses, or may be investigated independently.

ART. 58. PROP. *To find the focus of the emergent rays when a small pencil of diverging rays is incident directly on a double-concave lens.*



Let $PAP'B$ be the lens, of which r is the radius of the first, and s the radius of the second surface. The point Q being the focus of the incident rays, q' of the refracted rays within the lens, and q of the emergent rays; let, as before, $QA = u$, $q'A = u'$, $qB = v$, and $AB = t$.

At the first refraction, by Art. 44,

$$\frac{\mu}{u'} = \frac{\mu - 1}{r} + \frac{1}{u}$$

At the second refraction, by Cor. Art. 46,

$$\frac{\mu}{Bq'} = -\frac{\mu - 1}{s} + \frac{1}{v}$$

If the thickness is so small that we may put $Bq' = u'$, subtracting we have

$$\frac{1}{v} = (\mu - 1)\left\{\frac{1}{r} + \frac{1}{s}\right\} + \frac{1}{u}$$

$$= \frac{1}{f} + \frac{1}{u}$$

If the thickness be small, but not negligible, we find, in the same way as before,

$$\frac{1}{v} = \frac{1}{f} + \frac{1}{u} - \frac{t}{\mu}\left(\frac{\mu}{u'^2}\right)^2$$

The expression $\frac{1}{v} = \frac{1}{f} + \frac{1}{u}$ shows us that v cannot change sign, or q fall on the other side of the lens, unless Q falls on the other side, and $-u$ is less than f. Therefore q is a virtual focus for all diverging and some converging pencils, but is real when the incident pencil converges to a point nearer the lens than the principal focal distance.

The other cases of converging pencils and different forms of lenses may be all solved in the same manner; or all the cases which can occur in the direct refraction of small pencils by lenses may be deduced from either of the two latter propositions by taking any of the quantities u, v, r, s, f, negative when measured in the contrary direction to what they have been in the standard case.

Taking the double-convex lens and a diverging pencil for our standard case, as in Article 57, we have the formula

$$\frac{1}{v}=(\mu-1)\left\{\frac{1}{r}+\frac{1}{s}\right\}-\frac{1}{u}$$

or $$\frac{1}{v}=\frac{1}{f}-\frac{1}{u}$$

ART. 59. *To adapt the standard formula to the case of parallel rays incident on a concave-meniscus.*

Since the incident rays are parallel, Q is at an infinite distance, or $u=\infty$ $\therefore \frac{1}{u}=0$; and since the first surface is concave $\therefore r$ is negative, but greater than s, because the lens is thickest in the middle; also q is the principal focus, or $v=f$.

$\therefore$ we have $$\frac{1}{v}=\frac{1}{f}=(\mu-1)\left\{\frac{1}{s}-\frac{1}{r}\right\}$$

ART. 60. *To adapt the standard formula to the cases of parallel rays incident on plano-concave and concavo-plane lenses.*

Here $\frac{1}{u}=0$, since the incident rays are parallel; the radius of the concave surface is negative, and the radius of the plane surface is infinite, or its reciprocal is zero; also $v=f$.

$\therefore$ for a plano-concave lens,

$$\frac{1}{v}=\frac{1}{f}=-\frac{\mu-1}{s}$$

and for a concavo-plane lens,

$$\frac{1}{v}=\frac{1}{f}=-\frac{\mu-1}{r}$$

The negative sign shows us that the principal focus is on the opposite side of the lens to what it is in the standard case.

ART. 61. *To adapt the standard formula to the case of parallel rays incident on a convexo-concave lens.*

Here $\frac{1}{u}=0$, and r is positive and s negative; but s less than r, since the lens is thinnest in the middle.

$$\therefore \frac{1}{v}=\frac{1}{f}=-(\mu-1)\left\{\frac{1}{s}-\frac{1}{r}\right\}$$

or the focal distance is negative, as in the other concave lenses.

Art. 62. *To adapt the standard formula to the case of a diverging pencil incident upon a double-concave lens*: or to deduce the proposition of Art. 58 from that of Art. 57.

Since the incident pencil is diverging, u remains as in the standard case; but both surfaces being concave, the signs of r and s must be taken negative.

$$\therefore \frac{1}{v}=-(\mu-1)\left\{\frac{1}{r}+\frac{1}{s}\right\}-\frac{1}{u}$$

which shews that v is negative, and must be measured in the opposite direction to the standard case.

Taking v in the contrary direction, and changing the signs on both sides of the equation, we have, as in Art. 58,

$$\frac{1}{v}=(\mu-1)\left\{\frac{1}{r}+\frac{1}{s}\right\}+\frac{1}{u}.$$

or,

$$\frac{1}{v}=\frac{1}{f}+\frac{1}{u}$$

Art. 63. *To adapt the standard formula to diverging pencils incident upon plano-convex and convexo-plane lenses.*

We suppose the focus of the incident rays to be beyond the principal focal length, or u greater than f. Taking the radius of the plane surface infinite, we have for the plano-convex lens

$$\frac{1}{v}=\frac{\mu-1}{s}-\frac{1}{u}$$

and for the convexo-plane lens

$$\frac{1}{v}=\frac{\mu-1}{r}-\frac{1}{u}$$

Both these expressions are contained in the general one,

$$\frac{1}{v}=\frac{1}{f}-\frac{1}{u}$$

when f has the value of the principal focal distance for each form respectively.

ART. 64. *To adapt the standard formula to diverging pencils incident upon plano-concave and concavo-plane lenses.*

If we take the formula $\frac{1}{v}=\frac{1}{f}-\frac{1}{u}$, we have, by Art. 60, f negative for these, as well as other concave lenses. Therefore the formula becomes

$$\frac{1}{v}=-\frac{1}{f}-\frac{1}{u}$$

and v must be negative; measuring it consequently in the opposite direction to the standard case,

$$-\frac{1}{v}=-\frac{1}{f}-\frac{1}{u}$$

or,

$$\frac{1}{v}=\frac{1}{f}+\frac{1}{u}$$

the same form as in Art. 62.

ART. 65. *To adapt the standard formula to the case of a convex lens, and the incident pencil diverging from a point nearer to the lens than the principal focal length.*

Taking the general formula $\frac{1}{v}=\frac{1}{f}-\frac{1}{u}$, we have now u less than f, and therefore $\frac{1}{u}$ greater than $\frac{1}{f}$, consequently $\frac{1}{v}=-\left(\frac{1}{u}-\frac{1}{f}\right)$ a negative quantity; which shows that the focus of the emergent rays is virtual, and q falls on the same side of the lens as Q: changing the sign of v, we have $-\frac{1}{v}=\frac{1}{f}-\frac{1}{u}$.

ART. 66. *To adapt the standard formula to the case of converging pencils incident on convex lenses.*

Since the pencils converge, or Q falls on the opposite side of the lens to the standard case, we must take u negative, and the formula becomes

$$\frac{1}{v}=\frac{1}{f}+\frac{1}{u}.$$

ART. 67. *To adapt the standard formula to converging pencils incident on concave lenses.*

In these cases we have both u and f negative, and

$$\frac{1}{v}=-\frac{1}{f}+\frac{1}{u}$$

which shows that there are two cases, as noticed at the end of Article 58.

ART. 68. *To trace the changes in the corresponding positions of the conjugate foci for a convex lens.*

Taking the formula $\frac{1}{v}=\frac{1}{f}-\frac{1}{u}$, which refers to the figure of Article 57, we see that

if $u=\infty$ then $\frac{1}{v}=\frac{1}{f}$ $\therefore v=f$ and

q coincides with F.

„ $u<\infty>f$ .... $\frac{1}{v}<\frac{1}{f}$ $\therefore v>f$ ...

q lies beyond F from the lens.

„ $u=2f$ .... $\frac{1}{v}=\frac{1}{2f}$ $\therefore v=2f$ ...

q and Q are equally distant from the lens.

„ $u=f$ .... $\frac{1}{v}=0$ $\therefore v=\infty$ ...

q lies at an infinite distance.

„ $u>0<f$ .... $\frac{1}{v}$ is $-^{ve}$ $\therefore v$ is $-^{ve}$ ...

q lies on the same side of the lens with Q.

if $u=0$ then $\frac{1}{v}=\infty$ $\therefore v=0$ and

q coincides with Q at the lens.

„ u is $-^{\text{ve}}$ .... $\frac{1}{v}$ is $+^{\text{ve}}$ $\therefore v$ is $+^{\text{ve}}$ ...

q lies on the original side of the lens.

„ $u=-\infty$ .... $\frac{1}{v}=\frac{1}{f}$ $\therefore v=f$ ...

q is at the principal focus.

We see that Q and q move in the same direction and coincide at the lens; on either side of this point they separate again. When the distance of Q from the lens is $2f$, the distance of q is the same on the opposite side; this is their minimum distance *when on opposite sides* of the lens.

Art. 69. Prop. *To express the distance between the conjugate foci in terms of the focal distance and the distance of either of them from the lens.*

Since $\frac{1}{v}=\frac{1}{f}-\frac{1}{u}$ $\therefore v=\frac{fu}{u-f}$

and $u+v=$ distance $Qq=u+\frac{fu}{u-f}$

$$=\frac{u^2}{u-f}$$

$$=\frac{v^2}{v-f}, \text{ by a similar procedure.}$$

Cor. $\frac{u^2}{v^2}=\frac{u-f}{v-f}$, or $\frac{u}{v}=\sqrt{\frac{u-f}{v-f}}$; that is, the distances of the conjugate foci from the lens are as the square roots of their distances from the principal foci on their respective sides of the lens.

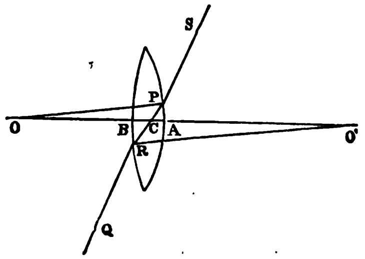
ON THE OBLIQUE REFRACTION OF SMALL PENCILS BY LENSES.

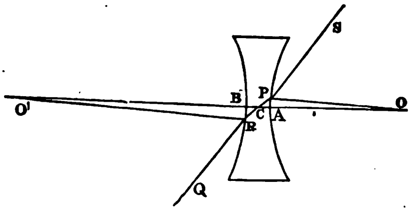
When a small pencil of rays falls obliquely on a refracting surface, there is generally *confusion*, as in mirrors: see the latter part of Chapter III.; but if the obliquity is small, the effect is so inconsiderable that it may be neglected in problems where we only take first approximations.

Definition. The *center of a lens* is the point where a line joining the extremities of two parallel radii of the surfaces cuts the axis of the lens.

Art. 70. Prop. *To shew that the center of a lens is a fixed point, and that it is* within *the double-convex and double-concave lenses;* upon the curved surface *of plano-convex and plano-concave lenses; and* without *the meniscus and concavo-convex lenses.*

Let O and O' be the centers of the first and second surfaces of the lenses in the figures; and let OP, $O'R$, be any pair of parallel radii. Let C be the point where the line joining the points P and R, or that line produced, meets the line OO', or the axis of the lens; then by definition, C is the center of the lens.





The triangles OCP, $O'CR$, are similar; and

$$\frac{OC}{OP}=\frac{O'C}{O'R}$$

Or, putting $OP=r$, $O'R=s$, $AB=t$, as before, we find in the double-convex and double-concave lenses

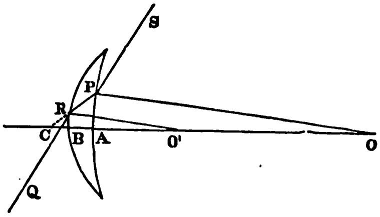
$$AC=\frac{tr}{r+s},\quad BC=\frac{ts}{r+s}$$

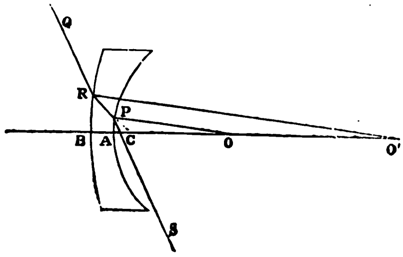
in the meniscus,

$$AC=\frac{tr}{r-s},\quad BC=\frac{ts}{r-s}$$

in the concavo-convex,

$$AC=\frac{tr}{s-r},\quad BC=\frac{ts}{s-r}$$





These expressions, as well as the figures, show us that C is within the lens in the two first forms, and without in the two last. The distances of C from A and B depend on the thickness of the lens and the radii of its surfaces, but are independent of any particular positions of the points P and R, and therefore are constant.

The positions of C for either of the latter might be deduced from either of the first forms, by taking the quantities negative which are measured in a contrary direction to what they are in the standard case.

Putting $r=\infty$ for a plano-convex or plano-concave lens, we have

$$AC=t,\ BC=0$$

or C coincides with B, the point where the axis of the lens meets the curved surface.

ART. 71. PROP. *The incident and emergent rays, corresponding to a ray within the lens whose direction passes through the center, are parallel; or a ray passing through a lens whose direction within the lens passes through the center has no deviation.*

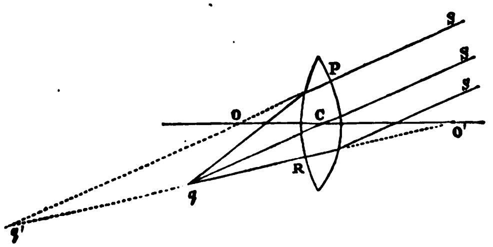
Let $SPRQ$ be the course of a ray in the figures of the last proposition. Since the radii OP, $O'R$, are parallel, the line PR, or the direction of the refracted ray within the lens, makes equal angles with them; and the angles of incidence and emergence are equal, whichever way the ray passes; and consequently SP, QR, are parallel.

We see that there is a lateral displacement, but no angular deviation of the emergent ray. The lateral displacement becomes very small for a thin lens when the obliquity is small; so that in such cases we may, without any sensible error, consider the incident and emergent rays to be in one straight line when their direction passes through the center; and hence the foci of small oblique pencils must be in this line.

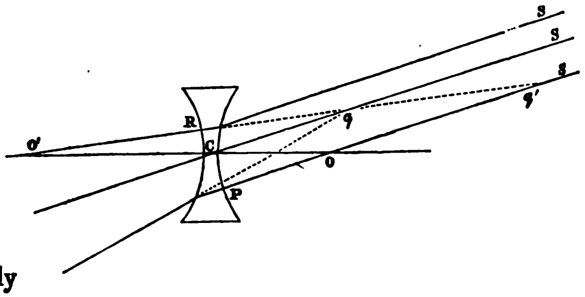
DEFINITION. The points where the directions of the incident and emergent rays cut the axis of the lens are called *the* *focal centers.* If D be the point where SP cuts OO', E that where QR cuts it, we find $AD=\frac{AC}{\mu}$, $BE=\frac{BC}{\mu}$ nearly for all the lenses.

ART. 72. PROP. *To find the focus of the emergent rays when a small pencil of parallel rays is incident obliquely upon a thin double-convex or double-concave lens.*

Let O and O' be the centers of the first and second surfaces respectively, and C the center of the lens.



Let SC be the ray of the pencil which passes through the center C, and emerges without deviation; and SP the ray which falls perpendicularly on the first surface, so that OP is its direction in the lens. The focus q' of the rays within the lens is in this line. Draw the line $q'RO'$; this will be the direction of the ray, which passes perpendicularly through the second surface, and therefore the focus of the emergent rays will be in this line; and being also in the line SC, will be at their intersection q.



To find Cq, the distance of the focus from the center of the lens, we have, as in Arts. 51 and 54,

$$\frac{\mu}{Pq'}=\frac{\mu-1}{r}$$

At the second refraction,

$$-\frac{\mu}{Rq'}=\frac{\mu-1}{s}-\frac{1}{Rq}$$

Since the pencil is supposed small and the lens thin compared with the radii of the surfaces, we may put $Pq' = Rq'$, and $Rq = Cq$.

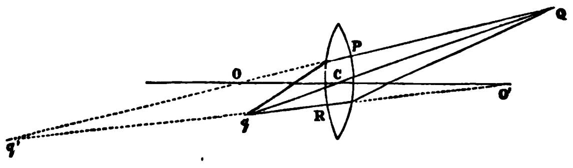
$$\begin{aligned}\therefore \frac{1}{Cq} &= (\mu-1)\left\{\frac{1}{r}+\frac{1}{s}\right\}\\ &= \frac{1}{f}\end{aligned}$$

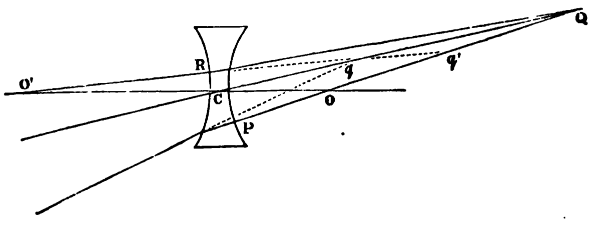
Or the principal foci, for small pencils but little oblique to the axis of the lens, are in a circular arc, of which the center is the center of the lens, and the radius the distance of the principal focus for the direct pencil, from that point.

COR. The same method will apply with the same result to the meniscus and concavo-convex, when the center is not far from the lens; and also, with little modification, to plano-convex and plano-concave lenses.

Conversely; if the pencil had diverged from q in the first figure, or converged towards q in the second, the emergent pencil would have consisted of parallel rays.

ART. 73. PROP. *To find the conjugate focus when a small pencil of diverging rays is incident obliquely on a double-convex or a double-concave lens.*





Let Q be the focus of the incident rays; O and O' the centers of the first and second surfaces respectively.

Let QP be the ray whose direction passes through O, and therefore enters the lens without deviation; the focus q' of the refracted rays within the lens will be in this line. Join q', O'; the ray within the lens which has this direction will emerge at R without deviation, and therefore the focus of the emergent pencil will be in this line; but it is also in the line QC passing through the center of the lens, and is consequently at their intersection q.

To find the distance qC, using the previous notation, we have, as in Arts. 57 and 58,

$$\frac{\mu}{q'P} = \frac{\mu-1}{r} \mp \frac{1}{QP}$$

and at the second refraction,

$$-\frac{\mu}{q'R} = \frac{\mu-1}{s} - \frac{1}{qR}$$

Or, since the pencil is small and the lens thin, we may put

$$q'P = q'R, \quad QP = QC, \quad qR = qC$$

$$\therefore \frac{1}{qC} = (\mu-1)\left\{\frac{1}{r} + \frac{1}{s}\right\} \mp \frac{1}{QC}$$

This expression is of the same form as that found in the respective cases for direct pencils. Let

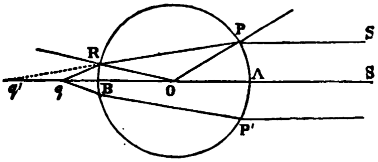
$$qC = v_2, \quad QC = u_2,$$

we have

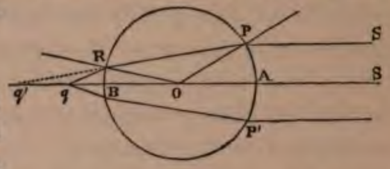
$$\frac{1}{v_2} = \frac{1}{f} \mp \frac{1}{u_2}$$

which may be adapted to all other cases, of converging pencils and other forms of lenses, as was done with regard to the formula for direct pencils.

ART. 74. PROP. *To find the principal focus of a refracting sphere.*



Let O be the center of the sphere, of which the radius is r. Let $SAOB$ be the ray which passes through it without deviation; SP a ray incident at P, and refracted in PR, of which the direction meets $SAOB$ in q', the focus of the rays after the first refraction; and let the emergent ray Rq meet SO in q, the focus of the emergent rays.



The triangle POR being isosceles, the angles at P and R are equal; or in a sphere the angle of refraction at the first surface equals the angle of incidence upon the second; and hence the angle of incidence always equals the angle of emergence.

In the triangles $q'PO$, $q'Rq$,

$$\begin{aligned}\angle Pq'O &= \angle POA - \angle q'PO \\ &= \text{angle of incidence} - \text{angle of refraction} \\ &= \text{angle of emergence} - \text{angle of incidence on the second surface} \\ &= \angle q'Rq;\end{aligned}$$

therefore the triangle $q'Rq$ is isosceles, or $q'q = qR$ and ultimately $q'B = 2.qB$.

Adding $AB = 2BO$ to each side, we have

$$q'B + AB = 2BO + 2qB$$

or

$$\begin{aligned}q'A &= 2qO \\ &= \frac{\mu r}{\mu - 1} \text{ from Art. 40;}\end{aligned}$$

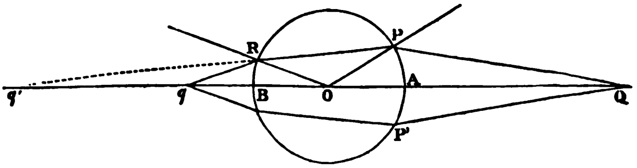
$$\therefore qO = \frac{\mu r}{2(\mu - 1)}$$

If $\mu = \frac{3}{2}$ for glass, $qO = \frac{3}{2}r$, or the principal focus lies half the radius beyond the surface; for water and $\mu = \frac{4}{3}$, $qO = 2r$.

If $\mu = 2$, $qO = r$, and q coincides with B; and if μ is less than 2, the principal focus is exterior to the sphere; if greater than 2, it is within it.

ART. 75. PROP. *To find the relation of the distances of the conjugate foci from the center when a small diverging pencil is incident directly on a sphere.*

Let Q be the focus of the incident diverging pencil, q' the focus of the refracted rays within the sphere, and q the focus of the emergent rays. Let $AO = r$, $QO = p$, $q'O = q'$, $qO = q$.



From the triangles QPO and $q'PO$, we have, as in Art. 43,

$$\frac{QA}{QO}=\frac{1}{\mu}\cdot\frac{q'A}{q'O}$$

or

$$\frac{p-r}{p}=\frac{1}{\mu}\left(\frac{q'+r}{q'}\right)$$

or

$$\frac{1}{r}-\frac{1}{p}=\frac{1}{\mu r}+\frac{1}{\mu q'}$$

At the second refraction similarly

$$\frac{qB}{qO}=\frac{1}{\mu}\cdot\frac{q'B}{q'O}$$

or

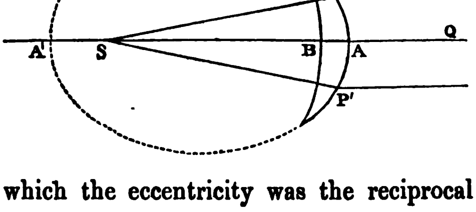
$$\frac{1}{r}-\frac{1}{q}=\frac{1}{\mu r}-\frac{1}{\mu q'}$$

Adding to the former, we find

$$\frac{1}{q}=\frac{2(\mu-1)}{\mu r}-\frac{1}{p}$$

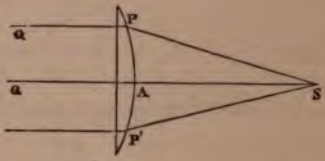
$=\frac{1}{f}-\frac{1}{p}$, if we put f equal the principal focal length.

Art. 76. Prop. *To find the figures of the lenses of accurate refraction for large direct pencils.*



When the incident pencil consists of parallel rays, it was shewn in Art. 37, that if the first surface PAP' was a prolate spheroid formed by the revolution of an ellipse about AA' its major axis, of which the eccentricity was the reciprocal of the refractive index; then the refracted pencil within the medium would converge accurately to the further focus S. If with center S and radius SB we describe a spherical surface and form the convex-meniscus $PAP'B$, every ray of the converging pencil will fall upon the second surface perpendicularly, and after emergence the whole pencil will accurately converge to S.

Reciprocally; if a pencil diverge from S and fall upon the concave-meniscus $PAP'B$, then the emergent pencil would consist accurately of parallel rays.



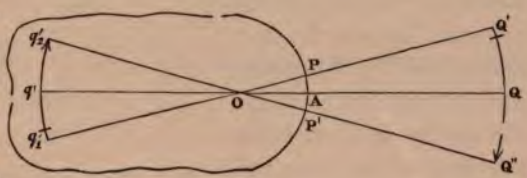
When a diverging pencil falls upon a convex surface which is an hyperboloid of revolution, the eccentricity of the generating hyperbola being equal to the refractive index, if the point from which the pencil diverges be the further focus of the hyperbola, then the rays within the medium will be all parallel to the axis of the hyperboloid, as shown in Art. 38. If the second surface of the lens be a plane perpendicular to the axis, every ray of the emergent pencil will pass through it without deviation, and will therefore consist of parallel rays.

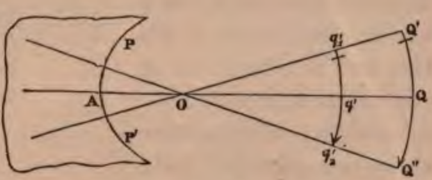
Reciprocally; if a pencil of parallel rays falls directly on a plano-convex lens of which the second surface is an hyperboloid of revolution of the eccentricity as above, the emergent pencil will converge to the further focus of the generating hyperbola.

In the cases of accurate refraction at spherical surfaces, as investigated in Art. 39, we might have lenses formed of which the second surface was spherical, with the center at the focus of the refracted rays, and the emergent pencil would diverge accurately from, or converge accurately to, that point. The figures of Art. 39 would thus give a concavo-convex, and a convex meniscus, respectively.

ON THE IMAGES FORMED BY SPHERICAL SURFACES OF MEDIA, LENSES, AND SPHERES.

Art. 77. Prop. *To find the image of an object produced by a spherical surface of a medium, the object being a concentric circular arc.*



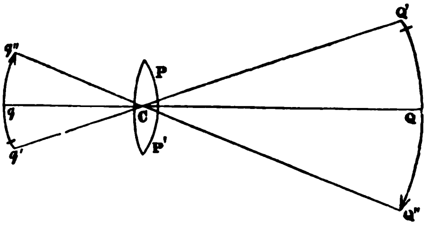


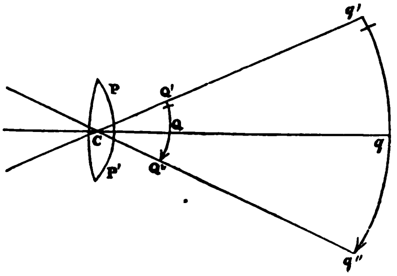
Let O be the center of the spherical surface PAP' of the medium, which is convex in the upper and concave in the lower figure; and let $Q'\,Q\,Q''$ be the object, which is a circular arc with center O. If we take any points Q', Q, Q'', and draw lines from them through O, the respective conjugate foci q'_1, q', q'_2 will be, by Arts. 43 and 44, in these lines as in the figures; and the object being everywhere equally distant from O, the various conjugate foci which make up the image will be so also, and the image will be a concentric arc $q'_1\,q'\,q'_2$ similar to the object.

The image is real and inverted in the first figure, and virtual and erect in the second.

The other cases, for other distances of the object from the surface, may be investigated in the same manner; and pencils which before their incidence on the refracting surface were converging to form an image, will give refracted pencils forming a conjugate image, whose form and position will be easily traced by means of Arts. 45 and 46. When the object is at an indefinitely great distance, the image is at the distance of the principal focus from O; and as the object approaches nearer, the image moves in the same direction; the two coinciding at O in the lower figure, and at the surface in both figures, as seen from Art. 50.

Art. 78. Prop. *To find the form of the image of an object which is a concentric circular arc produced by a convex lens.*





Let PCP' be the convex lens, and let $Q'\,Q\,Q''$ be the object, which is a circular arc with center C. Let $Q'QQ''$ be beyond the principal focus in the first, and within it in the second figure; the line through the points Q, C, q, being the axis of the lens, and Q, q conjugate foci; then q will be a real focus in the first and a virtual one in the second figure, by Arts. 57 and 65; and we have, if $f=$ the principal focal length,

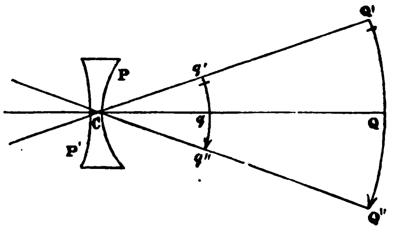
$$\pm\frac{1}{Cq}=\frac{1}{f}-\frac{1}{CQ}$$

If we take any other points Q' and Q'' in the object, and draw lines through C and these points, we shall have the conjugate foci q' and q'' equally distant from C with the point q, by Art. 73; or the image $q'qq''$ is a circular arc, having C for its center; and the image always subtends the same angle at the center of the lens which the object itself does.

In the first case, as in the figure, the image is always inverted; and in the second case, it is always erect: the magnitude of the image is to the magnitude of the object in the ratio Cq to CQ; and when real, may be either greater, equal to, or less than, the object; but when virtual, is always greater, except when the image and object coincide at the lens, and then are equal.

The image of a small straight line placed directly before the lens may be considered as a straight line similar to it, erect or inverted, magnified or diminished; but the real image of a straight line of considerable length will be more curved than that of a circular arc, since for points Q' and Q'' distant from the axis of the lens, the distances CQ' and CQ'' increasing, the formula gives Cq' and Cq'' diminishing. For this reason, the real image of a flat object cannot be received distinctly on a flat screen; but to be seen on a screen most distinctly, the screen must have the same curvature as the image. See PART II. page 136.

ART. 79. PROP. *To investigate the form and position of the image produced by a concave lens of an object which is a concentric circular arc.*



Let PCP' be the concave lens, and $Q'QQ''$ the object, which is a circular arc with center C; QC being the axis of the lens.

The focus of the emergent rays q, which is conjugate to Q, is given by Art. 58 from the formula

$$\frac{1}{Cq}=\frac{1}{f}+\frac{1}{CQ}$$

where we see that Cq must be always less than CQ.

Taking other points Q' and Q'' in the object, their conjugate foci will be at q' and q'', points similarly situated in a circular arc $q'qq''$, with center C and radius Cq.

The image is evidently similar to the object, erect, virtual, and diminished.

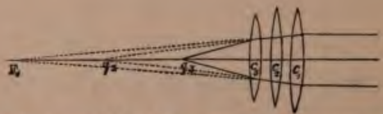
The figures in the two last propositions have been taken double convex and double concave; but the same method applies to the other forms of convex and concave lenses respectively.

We have considered the object to be entirely in the plane of the figure; but it is evident that a small object of any form placed directly before a lens will have an image similar to it, to or from the different points of which the rays of the various pencils, after eemrgence, will converge or diverge respectively.

We see in Part II. that when the object is large, the different parts of the image bear different ratios in magnitude to the corresponding parts of the object, or the image is distorted.

The rays, after emerging from a lens, may fall on another lens, and an image of the first image will be formed in the same way. When the first lens is convex, and the object beyond the principal focus, the emergent rays will fall in a state of convergence on another lens placed near it, and the primary image will be virtual to the second lens. The secondary image, or that formed by the pencils which emerge from the second lens, will be real when the second lens is also convex; but may be either real or virtual when it is concave, as will be seen in the proposition for finding the focal length of a combination of lenses.

Art. 80. Prop. *To find the focal length of a number of thin lenses in contact, and the nature of the image produced by them.*



Let c_1 be the center of the first lens, on which a pencil of parallel rays falls directly, and F_1 its principal focus; let c_2 be the center of the second lens, on which the pencil converging to F_1 is incident, and q_2 the focus, to which the rays emerging from it converge; let c_3 be the center of the third lens, on which the pencil converging to q_2 falling, the emergent rays converge to q_3, which is the principal focus of the combination.

We suppose the lenses to be so thin and placed so close together, that we may neglect the distance c_1c_3, in comparison of the focal lengths. Let f_1, f_2, f_3 be the principal focal lengths of the lenses c_1, c_2, c_3, respectively, and let $F = c_3q_3 = c_1q_3$ nearly, the focal length of the combination.

From Art. 66 we have

$$\frac{1}{c_2 q_2} = \frac{1}{f_2} + \frac{1}{c_2 F_1}$$

$$= \frac{1}{f_2} + \frac{1}{f_1}$$

and

$$\frac{1}{c_3 q_3} = \frac{1}{f_3} + \frac{1}{c_3 q_2}$$

$$= \frac{1}{f_3} + \frac{1}{f_2} + \frac{1}{f_1} = \frac{1}{F}$$

or,

$$\frac{1}{F} = \Sigma\left(\frac{1}{f}\right)$$

which we may suppose continued to any number of lenses.

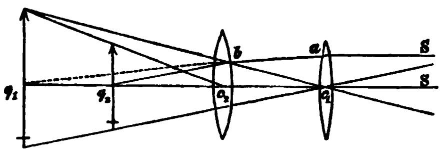
If any of the lenses were concave, their focal lengths would be negative, and in the summation must be taken with their proper signs; so that F being the focal length of the single lens, which is equivalent to the combination, it will be positive or negative, or will be the focal length of a convex or concave lens respectively, as the sum $\Sigma\left(\frac{1}{f}\right)$ is positive or negative.

If we call $\frac{1}{f}$ or the reciprocal of the focal length, *the power* of a lens, the proposition shows us that the power of a combination of any number of thin lenses in contact, equals the sum of the powers of the separate lenses. The image of an object formed by the combination will be the same in position and magnitude as would be formed by a lens whose focal length is F; and if u and v were the distances of conjugate foci, using the standard formula,

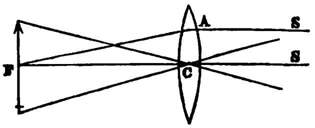
$$\frac{1}{v} = \frac{1}{F} - \frac{1}{u}$$

$$= \Sigma\left(\frac{1}{f}\right) - \frac{1}{u}$$

ART. 81. PROP. *To find the focal length of the lens, which is equivalent to two lenses separated by a given interval.*



Let C be the single lens, which is equivalent to the two c_1 and c_2, whose distance is d.



The lens C must have the same effect in causing a small direct pencil of parallel rays to converge, which the combination has; or a ray falling upon it at the same distance from the axis must after emergence converge to the axis at the same angle which a like ray does after emergence from the second of the two lenses; that is, if q_1 be the principal focus of c_1, and q_2 the point to which the pencil finally converges, $Sabq_2$ being the path of any ray, the angle bq_2c_2, at which the ray meets the axis, must equal the angle AFC, at which a like ray SAF meets the axis of the single lens.

Let f_1 and f_2 be the focal lengths of the lenses c_1 and c_2 respectively, F that of the lens C, and $AC = ac_1$ the distance of the like rays from the axis, which is very small; we must have

$$\frac{AC}{F} = \frac{bc_2}{c_2q_2}$$

but

$$\frac{bc_2}{q_1c_2} = \frac{ac_1}{c_1q_1} = \frac{AC}{f_1}$$

$$\therefore \frac{1}{F} = \frac{q_1c_2}{f_1 \cdot c_2q_2}$$

and

$$q_1c_2 = f_1 - d$$

$$\frac{1}{c_2q_2} = \frac{1}{f_2} + \frac{1}{f_1 - d} \text{ by Art. 66}$$

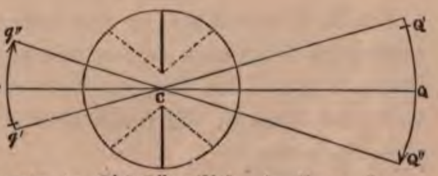
$$\therefore \frac{1}{F} = \frac{f_1 - d}{f_1}\left(\frac{1}{f_2} + \frac{1}{f_1 - d}\right) = \frac{f_1 + f_2 - d}{f_1 f_2}$$

We see that the power of the combination diminishes as the distance of the two lenses is increased, and that it is greatest when they are in contact. This formula is of great use to enable us to find the magnifying power of an optical instrument with a double eye-piece, when its power with a single eye-glass has been determined.

The same result is found when we make the condition of equivalence to be, that the combination of two lenses and the single one give equal images of a distant small object. The image at F in the lower figure subtends the same angle at C that the object does, which is also the same angle that the primary image at q_1 in the upper figure subtends at c_1.

Again, the secondary image at q_2 subtends the same angle at c_2 which the primary at q_1 does; and we must also have the magnitude of the image at q_2 equal to that of the image at F. From these conditions the formula above is easily obtained.

ART. 82. PROP. *To find the form and position of the image of an object which is a concentric circular arc, formed by a refracting sphere.*



Let C be the center of the sphere, and $Q'QQ''$ the object, which is a circular arc, whose center is also the center of the sphere; the foci $q'qq''$ conjugate to $Q'QQ''$ will be in lines drawn from those points through C, and all equally distant from it, or in a circular arc $q'qq''$, which is similar to $Q'QQ''$.

The image is real and inverted; and for the ratio of its magnitude to that of the object we have

$$\frac{q'q''}{Q'Q''} = \frac{qC}{QC}$$

The sphere has an advantage over a convex lens in having each pencil direct, however large the object may be; and these direct pencils may be made to be all small, by having a groove cut round the sphere in a plane passing through the center, as represented in the figure by the dark lines drawn from the circumference towards C; or by having the portion within the angle between the dotted lines ground away. Excellent pocket-magnifiers of the latter form are sold in the optician's shops, under the name of Coddington's lenses.

When the object is placed nearer to the sphere than its principal focus, the image, as in a convex lens, is virtual, erect, and magnified.

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EXAMPLES ON CHAPTERS V. AND VI.

Ex. 1. If a pencil of diverging rays incident on a convex surface of a medium is refracted to a focus equally distant from the surface with the focus of the incident rays, but on the opposite side of it, then the radius of the surface equals $\frac{\mu-1}{2(\mu+1)} \times$ distance between the conjugate foci.

Ex. 2. If the focus of the refracted rays is twice the distance from the convex surface of a medium with that of diverging incident rays, when they are both on the same side of it; shew that the focus of the incident rays is distant from the surface $\frac{(2-\mu) \times \text{radius}}{2(\mu-1)}$.

Ex. 3. Investigate directly the formula for a converging pencil incident on a concave refracting spherical surface; and shew that there will be two cases.

Ex. 4. Assuming the formula for a diverging pencil incident on a concave spherical refracting surface, deduce from it the two cases of diverging pencils incident on a convex surface.

Ex. 5. From any case of a diverging pencil incident on a spherical surface, deduce the formula for a pencil of rays diverging from a point within the medium, and emerging directly from a concave surface.

Ex. 6. From the result of the last question, deduce the relation of the apparent depth of a point seen directly below the surface of a fluid, compared with its real depth.

Ex. 7. Shew that the result for a pencil diverging from a distant point, and incident upon a convex surface of a dense medium, can be obtained by taking the pencil as coming the contrary way, and incident on a concave surface of a rare medium, with the refractive index $\frac{1}{\mu}$ from the dense into the rare medium.

Ex. 8. Trace the corresponding changes of position of the conjugate foci for a concave refracting spherical surface.

Ex. 9. Investigate directly the distance of the focus of the incident rays, when a small pencil passes directly through a double convex lens, and the emergent rays are parallel.

Ex. 10. Prove directly the formula $\frac{1}{v} = \frac{\mu - 1}{r} + \frac{1}{u}$ for a concavo-plane lens and diverging incident pencil.

Ex. 11. Prove the formula $\frac{1}{v} = (\mu - 1)\left\{\frac{1}{r} - \frac{1}{s}\right\} + \frac{1}{u}$ for a converging pencil refracted by a convex meniscus.

Ex. 12. Find directly the conjugate focus when a small pencil of diverging rays is incident obliquely on a plano-convex and a concavo-plane lens.

Ex. 13. Prove directly that the center of a plano-convex lens is upon the curved surface.

Ex. 14. When an object is placed before a convex lens at $1\frac{1}{2}$ times the focal length from it, show that the image is twice the linear dimension of the object; and if the object is at three times the focal length from the lens, the image is one-half the linear dimension of it.

Ex. 15. Shew that the image of the sun formed in the focus of a burning-lens has its diameter directly proportional to the focal length of the lens.

Ex. 16. The image of a heavenly body being formed in the principal focus of a convex lens, a plane mirror is placed perpendicular to the axis $1\frac{1}{2}$ times the distance of the image from the lens; shew that a second image will be formed by the reflected light on the opposite side of the lens, of the same linear dimension as the first.

Ex. 17. Investigate the form and position of the image when an object is placed nearer to a refracting sphere than the principal focus.

Ex. 18. If two thin lenses, the one convex, and the other concave, be placed close together, and the focal length of the concave is twice that of the convex; shew that the focal length of the combination is twice that of the convex lens.

Ex. 19. Shew that two convex lenses, of one inch focus each, separated by a distance of $\frac{1}{2}$ an inch, are equivalent to a single lens of $\frac{2}{3}$ of an inch focus; and if close together, they are equivalent to a single lens of $\frac{1}{2}$ inch focal length.

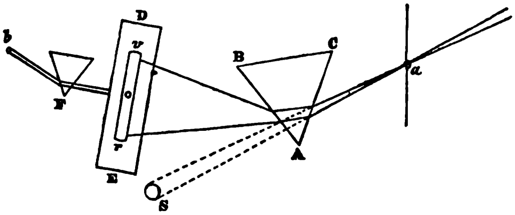
Ex. 20. A meniscus, of which the radii of the surfaces are 1 and 2, is placed in contact with a concavo-convex, of which the radii of the surfaces are 1 and 3; shew that the combination is equivalent to a concave lens of which the focal length is $\frac{6}{\mu-1}$.

CHAPTER VII.

ON CHROMATICS.

In the Introductory Chapter it was stated that the refractive index is different for different colours of light with the same medium.

The analysis of a beam of the sun's light by a prism was the experiment by which Sir Isaac Newton demonstrated his great optical discovery of the unequal refrangibility of the variously coloured rays.



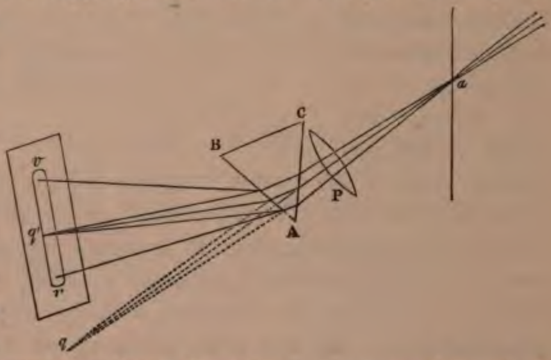
If we let a beam of the sun's light enter a dark room through a small hole in the window-shutter, as at *a* in the figure, and whilst it remains uninterrupted, if we receive it upon a screen, such as a piece of white card, we have a bright circle *S*, which subtends at *a* about 32 minutes of a degree, which is the sun's apparent diameter. If now we interrupt the beam by a prism as *ABC*, we have the direction of it changed by the refraction of the prism; but instead of a luminous circle, like that at *S*, which, by the deviation produced by the prism, is seen in a new position, we have, upon a screen *DE*, an oblong spectrum *rv*, of which the length is many times the breadth, and which is coloured red at *r* the lowest point, and violet at *v* the highest; the intermediate parts shading gradually from red, through orange, yellow, green, blue, indigo, to violet.

The prism must be turned round its edge backwards and forwards, until the spectrum *rv* is got into its lowest position, when the deviation of the middle rays is a minimum; and they make equal angles with the prism at incidence and emergence.

If through a small hole in *DE* we allow a stream of the coloured light to fall upon another prism as *F*, we find that it undergoes a deviation in direction, but is not at *b* again extended into an oblong spectrum; for its colour and figure remain very nearly as they would if they had not passed through the prism *F*.

From this we conclude that the spectrum *rv* has its lengthened form from the different deviation of differently coloured rays; the red rays having the least deviation and the least value of the refractive index, and the violet rays the greatest deviation and the greatest value of μ.

The spectrum *rv*, produced in the manner just described, has never the colours completely separated; for we must consider it as an assemblage of circles of the magnitude of *S*, of each different shade of colour overlapping each other. To produce a purer spectrum, we must have a narrow line of light at *a*, as, for instance, the light of the sun shining between the edges of two knives, each parallel to the edge of the prism *ABC*, and very near together: by placing a convex lens at *P* directly in the light, we have images of the line at *a* produced in each shade of colour at the conjugate focus as *q* for each; but when the prism *ABC* is interposed in the position for minimum deviation, each of the images of the line at *a* having a different deviation, the spectrum *rv* is now made up of narrow lines as at q', where the rays of some particular shade are brought to a focus. By looking attentively at a spectrum produced in this manner, we



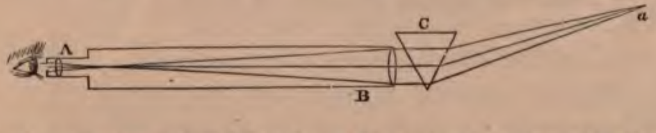
see that there are numerous dark bars across it, which are either deficient or much weaker shades in the gradual progression from red through the intermediate colours to violet.

The method of trying the experiment as in the figure was used by Sir Isaac Newton; and with a good prism and a lens of a few feet focal length, many lines can be seen thus on a screen of card; but after an imperfect notice of some of them by Dr. Wollaston, the full discovery of their great number and fixed positions in the solar spectrum was made by Fraunhofer with the fine prisms of the manufactory at Munich with which he was connected.

From their discoverer these dark lines in the spectrum are called *the fixed lines* of Fraunhofer; they exist in the light of the moon and planets as in that of the sun; but Fraunhofer discovered that in the light of the fixed stars they are different, and are also different in each different kind of artificial light.

By taking away the screen, and viewing the spectrum of the preceding experiment in the air by means of a magnifying eye-glass, an exceedingly great number of the fixed lines can be very well seen; but the method of experimenting with the prism attached to a telescope, which was used by Fraunhofer, is much more convenient.

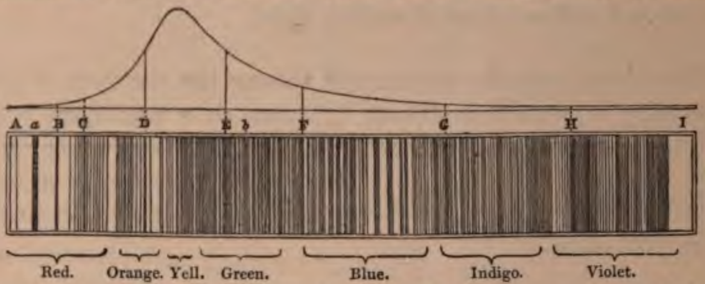
A telescope *AB*, which need not be achromatic, nor more than two feet long, but which must be supported on a convenient stand,



is to be set accurately to the focus for seeing the line of light at *a*, which should be as distant as possible, before the prism is put into its place.

The prism need not contain so much as a cubic inch of glass; but it must have its two refracting faces *accurately flat* and well polished, which is of more consequence than the glass being free from blebs and striæ: being attached to a support, fitting on the end of the telescope, so that the axis of the telescope is at the angle of minimum deviation with the prism for the green or middle rays of the spectrum, the telescope is to be turned so that the rays from the line of light at *a* fall upon the prism at the angle of minimum deviation in the same perpendicular section of the prism with the axis of the telescope; and then the Fraunhofer spectrum will be seen in the common focus of the eye-glass and object-glass. The plane of refraction *aBA* may be any that is convenient, as horizontal, vertical, or inclined; but in every case the line of light *must be parallel* to the edge of the prism, and the axis of the pencil must pass through it in a perpendicular section.

The annexed diagram may serve to give an idea of the appearance of some of the lines of the spectrum, and to show the positions of those which, after Fraunhofer's designation, are called



the lines *A*, *a*, *B*, *C*, *D*, *E*, *b*, *F*, *G*, *H*, of the spectrum: being each always in the same shade of colour, they serve to determine, with the greatest precision, the part to which a given refractive index belongs; and to shew the exact effect of different media in dispersing the different parts of the spectrum. The ordinates from the horizontal to the curve line at the upper part of the figure shew the intensity of the light at each part of the spectrum.

Sir Isaac Newton concluded, from insufficient experiments, that all media dispersed the colours of the spectrum in the same proportion as they refracted them; but Mr. Hall, and afterwards, about the year 1757, Mr. Dollond, discovered that in nearly all media the dispersion bears a different ratio to the refraction. It has since been recognized that every medium refracts the different parts of the spectrum in a peculiar manner, giving rise to what is called the irrationality of dispersion: thus, with a hollow prism, as mentioned after Art. 31, filled with essential oil of turpentine, it is found that the violet end of the spectrum is much more extended in proportion to the other parts than with a glass prism; and, on the contrary, if it is filled with sulphuric acid, that the red end is more extended in proportion to the other parts. Although in a less degree, the same holds for different kinds of glass, and offers a serious impediment to perfecting the achromatic telescope.

It was shown by Sir Isaac Newton that when the colours of the spectrum were compounded again, they reproduced white light; but the presence of all the colours is not necessary for this end, as any pair of complementary colours, or such as are taken from what may be called opposite parts of it when arranged as a circle, are sufficient: thus red, with an appropriate tint of green, produces white light, orange with blue, yellow with violet, &c.

In adopting a measure of the dispersion of media, we must take it in conjunction with the refraction; and the difference of the deviations for any two fixed rays of the spectrum, divided by the mean deviation, is called the *dispersive power* of the medium for those rays.

In Article 34 it was shown that if α be the refracting angle of a prism, μ the refractive index, d the deviation; then, when α is small, and the pencil passes nearly directly through the prism, we have

$$d=(\mu-1)\alpha$$

Let d_1 and μ_1 be the deviation and refractive index for some fixed ray of the spectrum, d_2 and μ_2 those for another fixed ray, and d and μ those for the ray which is the mean between them.

Then

$$\begin{aligned} d_1 &= (\mu\ -1)\alpha \\ d_2 &= (\mu_2-1)\alpha \\ d\ &= (\mu\ -1)\alpha \end{aligned}$$

Subtracting the first of these expressions from the second, and dividing the result by the third, we have

$$\frac{d_2-d_1}{d}=\frac{\mu_2-\mu_1}{\mu-1}$$

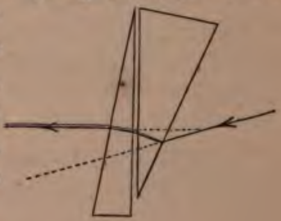
or, more concisely, we have the *dispersive power* of the medium for these rays $=\frac{\delta\mu}{\mu-1}$, for which we frequently write the letter p.

From what was said of the irrationality of the dispersion, it will be seen that this simple expression can never be more than an approximation; but if we suppose μ_1 and μ_2 to correspond to the fixed lines D and F of the spectrum in the nearly complementary colours orange and blue, comprising between them all the brightest part of the spectrum, the dispersive power so expressed will give results in the various propositions as accurate as can be applied in the construction of the usual optical instruments.

ART. 83. PROP. *To investigate the construction of an achromatic prism formed of two prisms of given media.*

By an achromatic prism, we mean one which produces a deviation of a pencil of white light, but does not separate the variously coloured rays.

Let d_1, d_2, μ_1, μ_2 be the deviations and refractive indices for the two fixed rays, and μ the mean refractive index in the prism, of which the refracting angle is α, and d'_1, d'_2, μ'_1, μ'_2, α', the same quantities respectively for the other prism. Since the dispersions of the prisms are to neutralize each other, they must be equal and opposite, and the refracting angles must be turned in opposite directions, as in the figure; that is, we must have



$$d_2-d_1=d'_2-d'_1$$

or,

$$(\mu_2-\mu_1)\alpha=(\mu'_2-\mu'_1)\alpha'$$

multiplying and dividing by $\mu-1$ and $\mu'-1$

$$\frac{\mu_2-\mu_1}{\mu-1}(\mu-1)\alpha=\frac{\mu'_2-\mu'_1}{\mu'-1}(\mu'-1)\alpha'$$

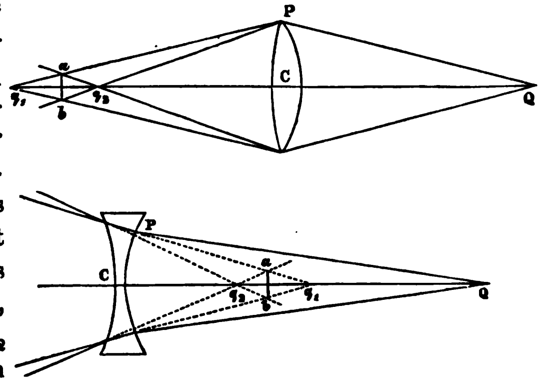
or,
$$\frac{\alpha}{\alpha'}=\frac{\dfrac{\delta\mu'}{\mu'-1}}{\dfrac{\delta\mu}{\mu-1}}\cdot\frac{\mu'-1}{\mu-1}$$
$$=\frac{p'(\mu'-1)}{p(\mu-1)}$$
if we put p and p' for the dispersive powers respectively.

From this formula the refracting angle of either one of the prisms can be found when that of the other and the refractive indices and dispersive powers are known.

It will be seen that there is always a small lateral displacement, although there is no angular separation of the colours.

If the angles of the prisms were large, we might obtain a theoretically more correct formula; but from what has been said of the irrationality of the spectrum, it will be seen that a perfectly achromatic prism is unattainable, with any kinds of glass now known.

Art. 84. Prop. *To find the longitudinal dispersion and the least circle of chromatic dispersion in a given lens.*



Let Q be the focus of the incident rays, q_1 that of the refracted rays, for which the refractive index is μ_1, and q_2 that for which it is μ_2; and $QC=u$, $q_1C=v_1$, $q_2C=v_2$ and μ the mean refractive index. If r and s be the radii of the first and second surfaces of the lens respectively, we have from Arts. 57 and 58,

$$\frac{1}{v_1}=(\mu_1-1)\left\{\frac{1}{r}+\frac{1}{s}\right\}\mp\frac{1}{u}$$
$$\frac{1}{v_2}=(\mu_2-1)\left\{\frac{1}{r}+\frac{1}{s}\right\}\mp\frac{1}{u}$$

Subtracting the first from the second,

$$\frac{1}{v_2}-\frac{1}{v_1}=(\mu_2-\mu_1)\left\{\frac{1}{r}+\frac{1}{s}\right\}$$

or,
$$\frac{v_1-v_2}{v_1v_2}=\frac{\mu_2-\mu_1}{\mu-1}(\mu-1)\left\{\frac{1}{r}+\frac{1}{s}\right\}$$
$$=\frac{\delta\mu}{\mu-1}\cdot\frac{1}{f}$$

Now v_1-v_2 is the decrease of the focal distance corresponding to the increase $\mu_2-\mu_1$ of the refractive index; therefore putting it $=-\delta v$, and p the dispersive power; also $v_1 . v_2=v^2$, we have $-\delta v=\frac{pv^2}{f}$, which is the longitudinal dispersion.

In the figures, ab, the diameter of the least circle, into which all the rays are collected, is nearly equally distant from q_1 and q_2, and is the diameter of the least circle of chromatic dispersion to be found.

Let $CP=y$, the half-aperture of the lens.

By similar triangles,

$$\frac{ab}{2}:y::\frac{q_1q_2}{2}:q_1C$$

or,
$$ab=\frac{y.\delta v}{v_1}$$
$$=\frac{pvy}{f} \text{ the diameter required.}$$

For distant objects and the incident rays parallel, we have $v=f$, and therefore the diameter of the least circle of chromatic dispersion $=ab=p.y$. This shews how a great advantage was obtained in long telescopes before the discovery of the achromatic telescope; since with the same eye-glass (see Art. 95) the magnifying power increases directly as the focal length of the object-glass, whilst the indistinctness arising from dispersion does not increase.

ART. 85. *To find the condition of achromatism in two thin lenses placed in contact.*

From Art. 80 we have $\frac{1}{v}=\Sigma\left(\frac{1}{f}\right)-\frac{1}{u}$; and the condition of achromatism is that v remains the same for all colours, that is, whilst the refractive indices are changed.

Let f and f' be the foci of the lenses, of which r and s, r' and s' are the radii of the surfaces, p and p' the dispersive powers respectively, and μ and μ' the mean refractive indices; we have, using the same notation as before,

$$\frac{1}{v}=(\mu_1-1)\left\{\frac{1}{r}+\frac{1}{s}\right\}+(\mu'_1-1)\left\{\frac{1}{r'}+\frac{1}{s'}\right\}-\frac{1}{u}$$

$$\frac{1}{v}=(\mu_2-1)\left\{\frac{1}{r}+\frac{1}{s}\right\}+(\mu'_2-1)\left\{\frac{1}{r'}+\frac{1}{s'}\right\}-\frac{1}{u}$$

subtracting,

$$0=(\mu_2-\mu_1)\left\{\frac{1}{r}+\frac{1}{s}\right\}+(\mu'_2-\mu'_1)\left\{\frac{1}{r'}+\frac{1}{s'}\right\}$$

$$=\frac{\delta\mu}{\mu-1}\cdot\frac{1}{f}+\frac{\delta\mu'}{\mu'-1}\cdot\frac{1}{f'}$$

or $$0=\frac{p}{f}+\frac{p'}{f'}$$

$\therefore$ the dispersive ratio $$\frac{p}{p'}=-\frac{f}{f'}$$

This equation shews us that one of the focal lengths must be negative and the other positive, since p and p' are always positive; or, one of the lenses must be convex and the other concave. Let f' be the focal length of the concave lens, and let F be the focal length of the combination;

then $$\frac{1}{F}=\frac{1}{f}-\frac{1}{f'}$$

And if the combination is to act as a convex lens, we must have F a positive quantity, and

$$\therefore \frac{1}{f}>\frac{1}{f'}$$

or $$f<f'$$

that is, the convex lens must be of the shortest focus or greatest power.

Also, $$0=\frac{p}{f}-\frac{p'}{f'}$$

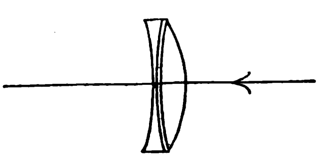
which shews us that p must be less than p', since f is less than f', or the concave lens must be of the higher dispersive power.

Putting $\frac{p}{p'}=\varpi=-\frac{f}{f'}$, we have

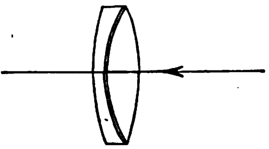
$$\frac{1}{F}=\frac{1-\varpi}{f}, \text{ or } f=F(1-\varpi) \text{ and } f'=F\left(\frac{1-\varpi}{\varpi}\right)$$

taking $\varpi=.6$ for some kinds of plate- and flint-glass, then for a thirty-inch focus telescope we have

$F=30$ inches, $f=12$ inches, and $f'=20$ inches.



An achromatic convex lens, therefore, may be as in the figures, and may be used to form real images of distant objects.



In practice, the convex lens is made of plate or crown glass, and the concave of the higher dispersing flint glass, in nearly all the object-glasses of the achromatic telescopes.

In one of the above figures the concave lens is a concavo-convex, but the practical opticians find the aberration most easily corrected when the concave lens is a concavo-concave, as in the other figure.

Art. 86. Prop. *To investigate the conditions of achromatism in a triple lens.*

Let f, f', f'' be the focal lengths of the three lenses; p, p', p'' the dispersive powers of the media of which they are composed. The general formula for a number of thin lenses in contact

$$\frac{1}{v}=\Sigma\left(\frac{1}{f}\right)-\frac{1}{u}$$

becomes $$\frac{1}{v}=\frac{1}{f}+\frac{1}{f'}+\frac{1}{f''}-\frac{1}{u}$$

Since v is to remain the same for rays of all colours, proceeding as in the last Article, we find,

$$0=\frac{p}{f}+\frac{p'}{f'}+\frac{p''}{f''} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

which shews us that either two of the lenses must be convex and the other concave, or otherwise two concave and the other convex. If one, as f', be negative, and the other two focal lengths be positive, that of the combination being F, we have

$$\frac{1}{F}=\frac{1}{f}-\frac{1}{f'}+\frac{1}{f''} \quad . \quad . \quad . \quad . \quad (2)$$

If the combination is to be essentially *convex*, we must have F positive; but the two equations (1) and (2) can be satisfied in an indefinite number of ways; so that the condition may be established, that other rays which in the double lens left a residual or secondary spectrum from the irrationality, may be now brought to one focus.

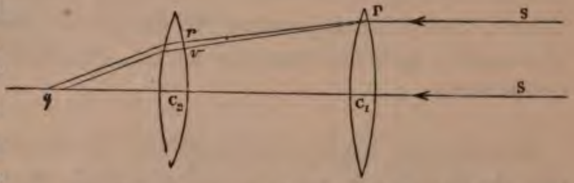
Although the triple object-glass has theoretically an advantage over the double one, yet it is now very rarely made for large telescopes, the additional errors of workmanship counterbalancing the small advantage to be gained. The double object-glass is found to shew minute objects, as small fixed stars, most distinctly when the secondary spectrum exhibits itself as a faint purplish or wine-coloured tint around the star.

In Art. 85 it was seen that the conditions of achromatism were fulfilled if the focal lengths and dispersive powers had a certain relation; but the focal lengths may remain the same, whilst the radii of the surfaces of the lenses are indefinitely changed; hence these radii may be so taken that the aberration of the convex lens shall be destroyed by the opposite aberration of the concave one; the lens is then said to be *aplanatic* as well as *achromatic*, and such a lens used as the object-glass of a telescope will bring a *large pencil* of incident rays very accurately to a focus.

Achromatic spheres and eye-lenses are easily made for common purposes by fixing two equi-convex lenses with a thin film of Canada balsam between the central parts of them. Canada balsam having the same refractive power as common glass, and nearly double the dispersive power, the concave of it between the two convex lenses is nearly sufficient to achromatise them; and the combination is much superior as a single magnifier to a single convex lens.

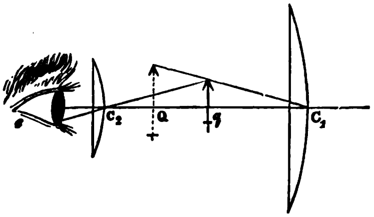
Art. 87. *To explain the principle of achromatic eye-pieces.*

The principle by which the achromatism in the eye-pieces of telescopes and microscopes is produced is very different to the above; the full investigation will be found at page 160, Part II., but an explanation of the way in which it takes place may be given here.



Let C_1 and C_2 be two convex lenses of the same kind of glass, having their axes in the same line SC_1C_2q. Let SP be a ray of white light incident on the first lens at P, parallel to the axis, but at a distance from it; by the prismatic effect of the lens, the red ray Pr will be less deviated than the violet one Pv; but Pv falling on a flatter part of the second lens than Pr, will be less deviated by it, and when the lenses are at a proper distance, a complete correction will result, and the rays emerging to q will be parallel, and shew no signs of colour. It is shewn in Part II. that this takes place when the lenses are separated a distance equal to half the sum of their focal lengths.

Since the distance and sum of the focal lengths are concerned only in the condition of achromatism, the forms of the lenses and the ratio of the focal lengths may be taken so as to secure other advantages; such as the diminution of the aberration and the flatness of the field of vision, as an eye-piece. The eye-pieces of Huygens and Ramsden are those most used in instruments where accuracy is a paramount object: the former, in telescopes and microscopes, to produce the most distinct vision, since it is achromatic and with very little aberration and distortion; and the latter, where a large extent of distinct field is desirable, as in the use of micrometers, but it is not quite achromatic, and is less distinct than Huygens's eye-piece.



Huygens's eye-piece consists of two convexo-plane lenses, as in the figure; the focal length of the more distant from the eye, which is called the field-glass, being three times that of the nearer or eye-glass; and their distance twice the focal length of the latter. The combination cannot, from the shortness of the focus of the eye-glass, be used to view an object; and the image, when used in a telescope or microscope, falls between the lenses, and hence it is called the negative eye-piece. In the figure, the image formed by the object-glass would have fallen at Q, but by the field-glass C_1, it is made to fall at q, half-way between the lenses, which is the principal focus of the eye-glass C_2; and the rays of each pencil on emerging from this latter are parallel, or as if they had come from a very distant object, and are therefore fit for vision by an eye at e.

If f_e be the focal length of the eye-glass C_2, then $3f_e$ is that of C_1, and

$$\frac{1}{C_1q}=\frac{1}{3f_e}+\frac{1}{C_1Q}$$

$$=\frac{1}{C_2q}$$

$$=\frac{1}{f_e}$$

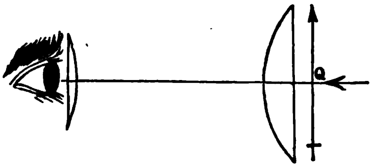
or,

$$C_1Q=\tfrac{3}{2}f_e$$

This is the distance at which the field-glass must be placed nearer to the object-glass than its focus, Q.

The lens C_1 is called the field-glass, from its having the effect of increasing the extent of the distinct part of the image or field of view.

Ramsden's eye-piece is formed of a plano-convex and a convexo-plane lens of equal focal lengths, set at a distance of about two-thirds of the focal length of either, as in the figure; and the image formed by the object-glass is immediately beyond the field-glass, as at *Q*, so that the pencils after passing through the eye-piece are fit for vision to an eye immediately behind the eye-lens. The wires of the micrometer (generally spider's lines), which are moved by the screws of the instrument, are also at the place of the arrow *Q*, and being seen with it are equally affected by the distortion of the eye-piece, and therefore the result of the measurement is unaffected.



If the lenses had been set at a distance equal to their focal length from each other, the condition of achromatism would have been fulfilled, but then the image *Q* must have coincided with the surface of the field-glass, and all dust and imperfection of the surface would have been seen with it, magnified by the eye-glass. The extent of distinct image is also increased by *Q* falling at some little distance beyond the field-glass.

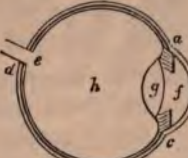
CHAPTER VIII.

ON THE EYE AND VISION.

THE eyes of man and the higher animals have many parts common to them and dioptrical instruments; but in the eye they are available in a wonderful manner, which no artificial instrument can be made to imitate.

The eyes of animals are far from uniform in their structure; and in many of the lower animals the absence of what we may call the optical parts leads to the conclusion that they serve to convey sensations of relative light and darkness only; whilst in the higher animals the presence of parts formed so as to produce images induces us to believe that they have perceptions of the forms of distant objects like ourselves: at the same time, the eyes of different tribes have peculiarities suited to their respective modes of living.

The human eye may be considered as composed of two segments of spheres; of which the front one is transparent, and called *the cornea, a b c* in the figure; and the hinder one, *a d c,* which is white, is called *the globe* of the eye. The length of the axis of the eye from *b* to *d* is rather less than an inch; and the radius of the cornea is about three-tenths of an inch.



The cornea is of a laminated structure, and of a considerable thickness, so that it is not injured by slight pressures on its surface: within its concavity is the aqueous humour *f,* contained in its appropriate membrane. Through the cornea is seen *the iris* or *uvea,* differently coloured in different individuals, of a fibrous structure, with the circular aperture in its center, called *the pupil.* The iris performs the important office of regulating the quantity of light which passes to the inner chambers, by its involuntary action enlarging and diminishing the diameter of the pupil. At a short space behind the iris is *the* *crystalline lens* or *humour g*, contained in its proper capsule. It is more convex at its posterior than its anterior surface, and is of variable density, being most dense at its center. Immediately in contact with the posterior surface of the lens is *the vitreous humour h*, which is contained in a delicate membrane, called the *hyaloid* membrane.

The optic nerve coming from the brain enters the globe of the eye rather nearer to the nose than the axis, as at *e*: after its entrance, it ramifies and forms *the retina*, or nervous membrane, which is immediately exterior to the posterior portion of the hyaloid and extends to the ciliary processes. Immediately exterior to the retina is *the membrana Jacobi*; and exterior to it again is *the choroid* coat or membrane, which contains the *nigrum pigmentum* in cells on its inner surface. The strong outer coat of the globe of the eye, called *the sclerotica*, is immediately exterior to the choroid, and is connected with the cornea at the points *a* and *c*. Near the junction of the sclerotica and cornea the choroid turns inwards in folds, called *the ciliary processes*, which, containing between them the nigrum pigmentum, form a dense screen, preventing irregular light from entering the vitreous humour and falling upon the retina. The *ciliary ligament* is a ring at the junction of the cornea and sclerotica, to the latter of which it is united as well as to the choroid.

The point *e* where the optic nerve enters, is insensible to the action of light, and is called *the punctum cæcum*. If two spots be made with ink on paper, a few inches apart and kept in the same horizontal line; then if we look steadily at the inner one with one eye, and move the paper nearer to and further from the eye, there will be one position at which all perception of the outer one will cease: this is when the image of it falls upon the punctum cæcum, where the optic nerve is not yet developed into the sensitive retina.

By dissecting away the sclerotica and choroid at the back of the eye, it is seen that inverted images are formed upon the retina, which soon becomes opaque after death, though transparent during life. We can have no doubt but that the sense of vision arises from the impression produced by these images upon the retina, which is conveyed by the optic nerve to the brain. If lenses be placed between the eye and an object, so that an indistinct image must be formed upon the retina, then vision is also indistinct.

The only part of an object which is seen absolutely distinctly is that of which the image falls upon the retina at the extremity of the axis of the eye, where there is a peculiar organization in the human eye. At this part is the *foramen centrale* of Sœmmering, which, however, is not an opening in the retina, but a small area, where the organization is more delicate.

The foramen centrale may be seen by most persons as existing in their own eyes, by means of the spectrum of Purkinje, or by means of that of Wheatstone*.

By holding a lighted candle about six or eight inches from the eye, and a little on one side, and agitating it, whilst we look with the eye into a dark space, most persons, after a short time, see before them numerous black, ramifying, waving lines, which represent the shadows of the blood-vessels of the retina, on a copper-coloured ground, with the foramen centrale represented in the middle of the spectrum. By the shadows caused by the candle being held on one side of the axis of vision, we see that the foramen centrale has generally a raised rim; and it is probable that the area within it is the place of most distinct vision; and from apparent change, when we are sensible of a change in the adjustment, it may be furnished with means of minute adjustment to distinct vision.

Wheatstone's spectrum is seen in the daylight by taking a piece of card about an inch square, with a hole in its center one-tenth of an inch in diameter, and agitating it before one eye turned to the sky, whilst the other is closed. The field of view becomes soon covered with an appearance of interwoven filaments, like threads of glass, except a small part in the center, which has a spotted appearance, and which is the foramen of Sœmmering. By agitating a lens of an inch focus, before the eye when looking towards the flame of a candle, we see larger waving lines in addition to the fine ones before mentioned, and conclude that the two spectra are only different forms of the same phenomenon; but that in Wheatstone's additional parts are seen.

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* This spectrum was first seen by Dr. Purkinje, but Mr. Wheatstone rediscovered it, not knowing that it had previously been discovered and published. The first spectrum was however described by Steinbuch in his 'Physiologie der Sinne,' published in 1811.

If a piece of burning wood be whirled quickly round in a circle in the dark, the whole circle appears luminous. This shews us that the impression of light upon the retina remains for some time, which is probably different in different eyes; and it is found to be in many eyes about one-seventh or eighth part of a second. By this property a person looking steadily at a bright object, and then turning his eyes suddenly to a dark surface, sees for an instant the appearance of the bright object upon it.

The eye's judgment of colour is very much influenced by the way in which it is affected by other colours or lights at the time. If a narrow strip of white paper be held near the face, and the eyes set so that two images are seen; then if a lighted candle be brought near one eye so as to shine into it, the two images, though from the same object, and illuminated by the same light, appear of different tints; the one seen by the eye not affected by the light appearing orange and brighter, and the other bluish and fainter. In the same way, when two shadows of a body are formed by different-coloured lights, if one of the lights be changed to another colour (as by putting another-coloured glass before it), the shadow on which the other-coloured light remained shining appears now of another colour. In the same way, also, painted colours appear much more intense when placed near their complementary or opposite colours. These effects are caused by the eye becoming less sensible of a weak colour when affected by a stronger of the same kind, and more sensible of it when affected by an opposite colour. From similar properties of the eye arise the following results: if we look steadily at a red wafer on a white ground, and then suddenly take it away, we soon see a green image in the place of the red one. One of the most singular effects produced by contrasted colours is when a red figure is worked in worsted on a light-blue ground, or a blue figure on a red ground; on being agitated in the light of a candle, the figure appears vibrating, as if it consisted of some jelly-like substance.

There are many cases on record of persons who have not had the ordinary perception of colours. Dr. Dalton, the father of the modern chemistry, had this peculiarity; and he has given an account of the most striking points connected with his vision, in a paper in the Memoirs of the Manchester Philosophical Society. He found that ordinary eyes saw nearly as he did, if they looked through a solution of sulphate of copper. The leaves and berries of the holly appear of exactly the same colour to such eyes. Pink and sky-blue are an exact match. Reds, dark-greens, and browns, are all called browns. Sir David Brewster's view, that the eyes which have this peculiarity are very little sensible to red light, will account for the above phenomena; for instance, pink is a mixture of pale red and pale blue, and the latter only is seen.

Many other properties of the eye and vision belong more properly to the science of physiology.

The refractive indices from air into the principal constituents of the eye were found by Sir David Brewster to be as follows:

For the aqueous humour . . . . . $\mu=1.336$
For the crystalline lens at its surface . $\mu=1.3767$
. . . . center . $\mu=1.3990$
. . . . the mean . $\mu=1.3839$
For the vitreous humour . . . . . $\mu=1.3394$

The relative refractive index for the aqueous humour into the crystalline lens, taking the mean value of μ, is found by Art. 27 to be 1.0358, and for the vitreous humour into the crystalline lens, 1.0332.

It will be seen that the principal refraction of a pencil takes place at its entering the cornea, and that the crystalline lens, lying between media of which the refractive powers are but little less than its own, has not much effect in shortening the focal length of the eye; but, being most dense at its center, the central rays of a pencil are more refracted than the extreme ones, and hence the spherical aberration of the cornea is corrected.

That the eye is achromatic is easily tested by looking at the smallest possible bright particle on a dark ground, when it is seen to be free from even the secondary spectra mentioned in Art. 86. We must be careful of testing its achromatism by experiments which introduce colours by diffraction (see Physical Optics) at the boundary of the iris or otherwise.

Ordinary eyes can see objects distinctly from about eight inches to indefinitely great distances; and since the image formed upon the retina must be in each case accurately at the focus, when there is distinct vision, it is clear that the eye possesses the means of adjusting itself to all distances between those limits. We are ourselves sensible that adjustment takes place when we turn our eye suddenly from one object at one distance to another at another distance; and we easily perceive that two objects in the same direction cannot be seen distinctly at the same time, if one is near and the other distant. The means by which the eye adjusts itself to distinct vision at different distances has been a subject of much discussion, and many hypotheses have been proposed which were entirely deficient of either optical or physiological foundation. The view of Dr. Pemberton, which was afterwards adopted by Dr. Young, supposes that the crystalline lens changes its form; but it is now known that it possesses no muscular structure or other apparatus which could change its form, independently of the inadmissibility of the magnitude of the changes either in form or position which could produce the adjustment required; for, its being imbedded in media of nearly the same refractive power would require the changes to be very considerable. The view of Dr. Jortin, that the cornea changes its convexity from the action of some muscular apparatus, seems to be the only tenable one. The refraction from the air into the cornea is so great, that a very slight change of convexity will produce the required adjustment for objects differing in distance from eight inches to infinity. When the change of adjustment is from a distant to a near object, the iris is always seen to contract, although the quantity of light falling upon the eye remains the same, and *vice versâ*; and this contraction is now known to physiologists to be caused by muscularity of the iris; so that we have undoubted evidence of muscular action accompanying the adjustment for different distances of the object viewed.

ON LONG AND SHORT-SIGHTEDNESS.

The eyes of every person have only a limited power of adjustment for distance; but the extremes of the distances for distinct vision are different for different persons.

Old persons generally lose the power of seeing near objects distinctly, whilst they retain it for distant objects; they are then said to be long-sighted or *presbyoptic.*

Many persons are able to see distinctly objects at the distance of two or three inches from the eye, but are not able to do so when they are removed to the distance of a few feet; such persons are called short-sighted or *myoptic.*

Art. 88. *To find the forms of the lenses required to remedy long and short-sightedness.*

In presbyopia the power of the eye is sufficient to bring the rays from distant objects to a focus upon the retina, as at q in fig. 1; but a pencil diverging from a near point, as Q, fig. 2, would have the conjugate focus as at q, and the rays would be spread over a considerable area upon the retina, and vision would be indistinct; if, however, we interpose a convex lens as at C, fig. 3, of which Q is the principal focus, then the rays falling in a state of parallelism upon the cornea, they will be brought to a focus q upon the retina,

Fig. 1.

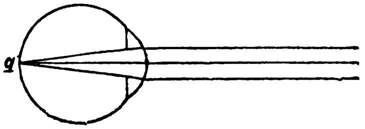


Fig. 2.

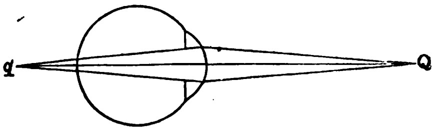
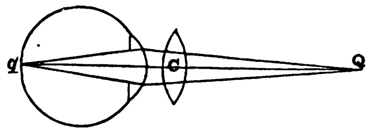


Fig. 3.



and vision will be distinct. In this manner a convex lens is the proper eye-lens to correct long-sightedness.

In myopia the power of the eye is so great that pencils diverging from near points, as *Q*, fig. 4, are brought to a focus *q* upon the retina, and vision is distinct for near objects; but the rays coming from a point in a distant object, as in fig. 5, are brought to a focus, as at *q*, before they reach the retina, and being spread over a considerable area on the retina vision is indistinct. By placing a concave lens, as at *C*, fig. 6, before the eye the rays may be made to fall in a state of divergence upon the cornea as if coming from a near point *Q*, fig. 6, and will then be brought to a focus *q* upon the retina, and vision will be distinct. In this manner a concave lens of proper focal length is the eye-glass requisite to correct short-sightedness.

Fig. 4.

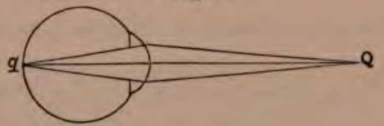


Fig. 5.

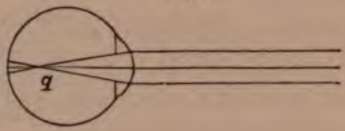
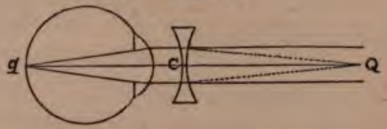


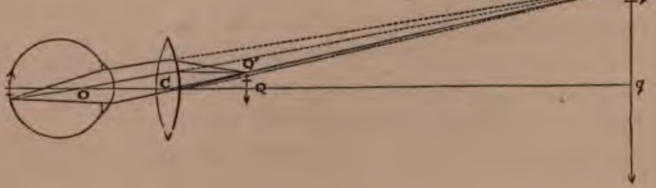
Fig. 6.



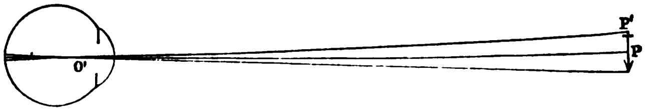
There are many persons short-sighted with one eye, but of ordinary sight with the other; a concave eye-glass for the short-sighted eye is a great relief to them.

Art. 89. *To find the conditions of distinct vision through a convex lens and its magnifying power.*

Let PP' be an object placed at the distance of distinct vision



before the eye in the figure below; and let C be a convex lens placed between the eye and the same object, QQ', in the figure



at the foot of the last page. When the vision is distinct in the case of the first figure, the rays must fall upon the cornea in the same state of divergence as in the second, when the eye remains in the same state of adjustment for distance; or, the rays must diverge from a virtual image qq', equally distant from the eye with PP', and they will then be brought to foci forming a distinct image upon the retina.

The magnifying power of the lens is the number of times the angle which $q'q$ subtends at the eye, contains the angle which PP' subtends. Let O and O' be the optical centers of the eye, which nearly coincide with the center of curvature of the cornea.

Then the magnifying power of the lens $= \dfrac{\angle q'Oq}{\angle P'O'P}$

$$= \frac{\tan q'Oq}{\tan P'O'P} \text{ nearly.}$$

$$= \frac{\;\dfrac{q'q}{qO}\;}{\;\dfrac{P'P}{O'P}\;}$$

$$= \frac{q'q}{P'P}$$

$$= \frac{q'q}{Q'Q}$$

$$= \frac{Cq}{CQ} \text{ from Art. 78.}$$

But, from Art. 65, if f be the focal length of the lens C, and we call Cq a positive quantity,

$$-\frac{1}{Cq} = \frac{1}{f} - \frac{1}{CQ}$$

or

$$\frac{1}{CQ} = \frac{1}{f} + \frac{1}{Cq}$$

Substituting, we have

$$\text{the magnifying power} = \frac{Cq}{f} + 1.$$

Let $e = O'P$, the eye's distance of distinct vision, and $d = OC$, the distance of the lens from the center of the eye,

$$\text{the magnifying power} = \frac{e-d}{f} + 1$$

$$= \frac{e}{f} - \frac{d}{f} + 1$$

which becomes greater as d is less, or as the lens is nearer to the eye. When $d = e$, the magnifying power $= 1$, or the effect of the lens ceases. When the focal length of the lens is small, and d also small compared with e, we have

$$\text{the magnifying power} = \frac{e}{f} \text{ nearly.}$$

This is the formula generally used in speaking of the magnifying power of a short-focused lens or *single microscope*. Let $f = \frac{1}{10}$ inch, and $e = 8$ inches, we have

$$\text{the magnifying power} = 80.$$

If we took $e = 10$ inches, then the magnifying power $= 100$.

In the case of myoptic vision, where e is small, as two or three inches, the magnifying power becomes small; but such eyes at their least distance of distinct vision see an object under a much greater angle than ordinary eyes, and have great advantages in the use of optical instruments, and in many optical experiments; the natural power of the short-sighted eye being much more accurate than can be produced by a lens for the ordinary eye.

CHAPTER IX.

ON OPTICAL INSTRUMENTS.

Of optical instruments we may form two classes; one of which consists of instruments for showing magnified images of objects, and the other of instruments in which optical combinations are applied to obtain measures of either the linear magnitude of objects or angles between given directions.

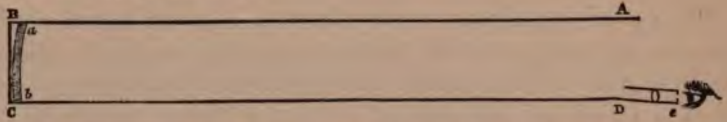
Although the welfare and progress of the human race are intimately connected with the advancement of every science, yet the benefits which have arisen, and are still arising, from the improvements in optical instruments seem second to no others. Astronomy, Geodesy, and Navigation, on which the intercourse of nations so much depends, would have been in a far different state if the optical parts of the respective instruments had remained rude and unimproved; and still more important, as coming to each individual in his own person, is the prospect of sound progress in the curative art, from the recent and daily discoveries which are arising in Physiology and Pathology by the state of perfection which the microscope has attained within the last few years.

In the first of the above-named classes, we have the catadioptrical instruments, the various forms of the reflecting telescopes and microscopes; the dioptrical instruments, the single and compound microscopes, the astronomical and the ordinary day-telescopes, the opera-glass, solar microscope, and magic-lantern, the camera-obscura, and the camera-lucida.

In the second class we have the various forms of micrometers, Hadley's sextant and reflecting circles, the optical square, the goniometer, and many instruments which are more properly explained in treatises on Astronomy.

THE HERSCHELIAN TELESCOPE.

The simplest telescope is the form of the reflecting telescope, which was adopted by Sir William Herschel, when very faint celestial objects were to be observed. The figure represents a section of the tube *ABCD*, of the speculum *ab*, and of the eye-tube and eye-glass at *e*.



The speculum *ab* is set obliquely to the axis of the tube, so that an image of the heavenly body, to which the axis of the tube is directed, is formed at the side where the eye-tube *e* is placed. This image is seen by the observer magnified by the eye-glass, as he stands at the front of the telescope with his back to the object. It is called *the front view.*

The image being formed by pencils reflected obliquely by the mirror, is never distinct when the aperture has the proportion to the focal length which is used in other reflecting telescopes; but when such faint objects as the smaller of the satellites of Saturn, those of Uranus, or faint nebulæ, are to be observed, this is of less consequence than the loss of light which occurs by the second reflexion in the other reflecting telescopes, which have two mirrors and an eye-glass. If the aperture is not more than one-fifteenth part of the focal length the confusion is not very great, and the advantages which are obtained by dispensing with the small mirror, required in other forms, render it the telescope that amateurs should first undertake to construct.

Supposing the mirror to reflect two-thirds of the light incident upon it, and the eye-glass to transmit about nine-tenths of the light falling upon it, we find sixty rays of every one hundred falling upon the mirror to reach the eye; and the larger achromatic refracting telescopes do not transmit more.

The magnifying power is found in the same way as for the

Newtonian telescope, and is the focal length of the speculum divided by that of the eye-glass, as in that form.

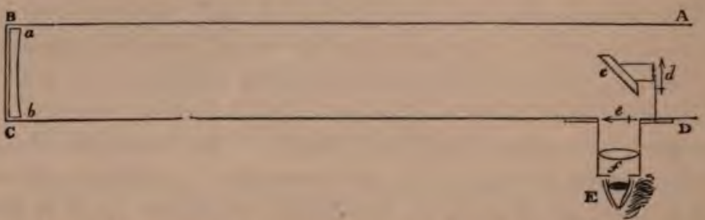
The quantity of light with which the image is illuminated, is proportional to the area of the mirror or the square of its aperture. The mirror of the large telescope of Sir William Herschel, intended for observing the faint celestial objects before named, was forty-eight inches in diameter of the aperture, and of forty feet focal length.

THE NEWTONIAN TELESCOPE.

This telescope has been termed the amateur's telescope, and in no-wise incorrectly. Sir Isaac Newton formed the plan, polished the mirrors, and fitted up the first reflecting telescopes ever made, which were found to be as effective as the refracting telescopes of the time, of twelve times their length. The celebrated John Hadley, about the year 1719, next succeeded in making a Newtonian telescope of about six feet length, which was equal to the refracting telescope given by Huygens to the Royal Society, of more than one hundred and twenty feet length. The Newtonian telescopes made by Sir William Herschel were the instruments with which he made his most important astronomical discoveries; his favourite working instrument being a seven-feet telescope of this form, with the large mirror of six and one-fourth inches aperture. The author may attribute his present position to his having in his youth taken the study and constructing of Newtonian reflecting telescopes and microscopes for his favourite amusement, leading him to other scientific and mathematical studies. He has the satisfaction to know that his labours were not lost; since the present noble, persevering, and spirited amateur maker of reflecting telescopes, the Earl of Rosse, says, in his paper in the Transactions of the Royal Society for 1840, though without acknowledgment, that he uses a chemically prepared powder for polishing, and a metallic chilling surface in the casting of his specula, as the author had published in Brewster's Edinburgh Journal of Science for 1831; the casting process, however, with a variation based upon the erroneous supposition that an air-bubble in a fluid has a tendency to escape downwards*.

The real image of a distant object formed by a concave mirror being, as shown in Art. 22, at the principal focus in front of the mirror, Sir Isaac Newton placed a plane mirror in the axis, and at an angle of 45° to it, between the large concave one and its focus, which reflected the pencils at right angles to their previous direction, so that the image was formed at the side of the tube of the telescope, and might there be viewed by an eye-glass.

In the figure, *ABCD* represents the tube of the telescope, *ab* the concave mirror, which, if the small mirror *c* were not interposed, would give an inverted image of a distant object in the direction of its axis, at *d*, in its principal focus; but the small plane mirror, *c* being set at an angle of 45° with the axis of *ab*, the image is formed at *e* in place of *d*, and in the focus of an eye-glass *f*, which is in a tube attached to a sliding piece, which also carries the plane mirror, and thus a magnified image is seen by an eye at *E*.



ART. 90. PROP. *To find the magnifying power and field of view of the Newtonian telescope.*

Since the only effect of the plane mirror is to reflect the pencils from the large mirror to the side of the tube, we may suppose it away, and the light to pass to the eye-glass directly in the axis of the telescope.

Let $A'AA''$ be the concave object-mirror, C the eye-glass, $q'qq''$

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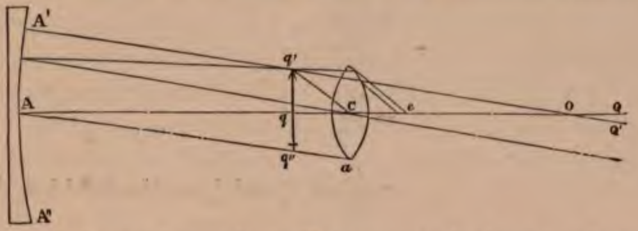
* See Philosophical Magazine for January 1850.

the inverted image of a distant object in the common focus of the object-speculum and eye-glass. We suppose the image to be small, and therefore the pencils diverging from it, and falling on the eye-lens, will, after emergence, become parallel in each pencil, and consequently fit for vision to ordinary eyes; as are also those from the distant object. The ray, as $q'C$ in the figure, which would pass through the center of the lens, would emerge without deviation, and therefore the other emergent rays must be parallel to this. Any rays reflected from the center of the aperture A, and falling on the eye-glass, will be refracted by it so as to cross the axis ACO in a point which is the conjugate focus to A; and since AC is always large compared with the focal length of the eye-lens, this point, as e in the figure, is nearly the principal focal distance from C. This is the place where the eye of the observer should be in order to receive all pencils which have been reflected centrically by the mirror, and the hole in the eye-tube of the telescope should be somewhat nearer to the lens than this.



If O be the center of curvature of the mirror, $QOqA$, $Q'Oq'A'$ the rays from different points in the object, which falling perpendicularly on the mirror are reflected directly back again, then the angle QOQ' which the points in the object subtend at O, is the same as the angle $q'Oq$, which the corresponding points in the image subtend; and the magnifying power of the telescope is the number of times the angle $q'Cq$, which the image subtends to the eye of the observer, contains the angle QOQ' or angle $q'Oq$.

$$\text{Or the magnifying power} = \frac{\angle q'Cq}{\angle q'Oq}$$

$$= \frac{\tan q'Cq}{\tan q'Oq} \text{ nearly, since } qq' \text{ is small.}$$

$$=\frac{\frac{qq'}{Cq}}{\frac{qq'}{Oq}}$$

$$=\frac{Oq}{Cq}$$

$$=\frac{f_o}{f_e},$$ if f_o and f_e are the focal lengths of the object-speculum and eye-lens respectively.

In finding the field of view, we see that if we join the center of the aperture A with the edge of the eye-lens by the line $Aq''a$, then of the full pencil from the mirror, which is brought to a focus at q'', the part from A to A'' will fall upon the eye-lens; but the part from A to A' will pass outside its edge: the point q'' will consequently be only seen by half pencils.

If we join A' and a, the point where the line cuts the image will be the boundary of the portion which is seen by full pencils or the brightest portion of the field of view. If we join A'' and a, this is the last ray of a pencil which can reach the eye-lens, and its intersection with the image gives the extreme of the field of view. The distance between the portion seen by full pencils and the extreme of the field of view is called the *ragged edge*, and is frequently partially or wholly cut off by *a stop* or *diaphragm*, consisting of a metal plate in the eye-tube, with a circular aperture of the required diameter in its center.

We may take the point q'', which is seen by half pencils, as limit of the field, the radius of which is qq'', and the angular radius to the naked eye is the angle $q''OA =$ angle $q''AO$

$$=\frac{aC}{AC}$$

Let a be the half-aperture of the eye-glass, then the angular radius of the field of view $=\frac{a}{f_o+f_e}$.

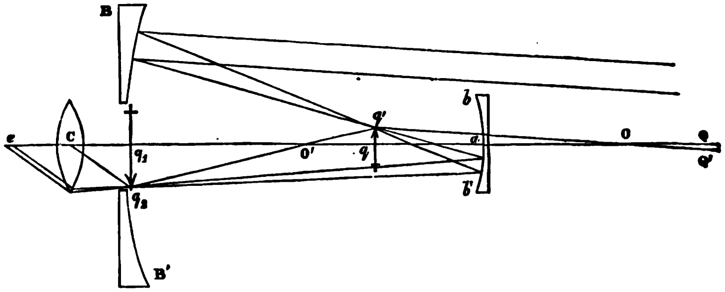
We here suppose the plane mirror to be sufficiently large to reflect all the required rays to the eye-glass.

A right-angled prism was proposed to be used in place of the small plane mirror by Sir Isaac Newton. The object was to save light by employing the total reflexion at the base of the prism. An achromatic prism to bend the pencils to the side of the tube at a convenient angle has been proposed by Sir David Brewster. The objections against these are, that additional errors of workmanship would be introduced by the increased number of surfaces; and refraction is always accompanied by aberration, which would more than counterbalance the greater loss of light by the metallic speculum.

ON THE GREGORIAN TELESCOPE.

This form of a reflecting telescope was the first invented; but its inventor, Dr. Gregory, was not able to get it executed. John Hadley, Esq., who was so successful with the Newtonian, appears afterwards to have constructed this form; and soon afterwards Mr. James Short brought the workmanship to such perfection that for a long series of years the Gregorian was the most effective telescope.

The Gregorian telescope consists of a large and a small concave mirror and an eye-lens, or more usually a double eye-piece. The large mirror has a circular hole through its center, through which the light reflected from the small mirror passes to the eye-piece.



Let *BB'* in the figure be the large or object-speculum, *bab'* the small one, and *C* a single eye-lens.

Let qq' be the inverted image of a distant object formed at the principal focus of the large mirror, and q_1q_2 the image of qq' formed by the small mirror in the focus of the eye-glass C. Then the pencils, which diverging from different points in q_1q_2 fall upon the eye-glass, will emerge from it in a state of parallelism, and therefore fit for vision to an eye placed at e, where they cross the axis of the telescope. The second image being erect, this telescope becomes a powerful one for day purposes, as well as an astronomical one.

ART. 91. PROP. *To find the magnifying power of the Gregorian telescope.*

Let O and O' be the centers of the curvatures of the two mirrors in the last figure. Let q' and q be the foci conjugate to the points Q and Q' in the distant object, and draw the line $q'OQ'$; qOQ being the axis of the telescope. Let q and q_1, q' and q_2 be conjugate foci in the two images, and draw the line $q'O'q_2$. Join C the center of the lens and q_2.

Then q_2Cq_1 is the angle under which the points q_2 and q_1 in the second image, corresponding to Q' and Q in the object, are seen by the eye at e; and the angle under which Q' and Q are seen by the naked eye is $Q'OQ = q'Oq$. Hence, putting f_o, f' and f_e for the focal lengths of the large and small specula and eye-glass respectively; the magnifying power

$$= \frac{\angle q_2Cq_1}{\angle q'Oq}$$

$$= \frac{\frac{q_1q_2}{Cq_1}}{\frac{q'q}{Oq}}$$

$$= \frac{q_1q_2}{q'q} \times \frac{f_o}{f_e}$$

$$= \frac{O'q_1}{O'q} \times \frac{f_o}{f_e} \text{ by Art. 22}$$

$$= \frac{aq_1}{aq} \times \frac{f_o}{f_e} \text{ by Art. 19}$$

Let $l = aq_1 =$ the distance between the centers of the apertures of the two mirors nearly, we have

$$\frac{1}{aq_1} = \frac{1}{f'} - \frac{1}{aq} \text{ from Art. 19}$$

or

$$\frac{1}{l} = \frac{1}{f'} - \frac{1}{aq}$$

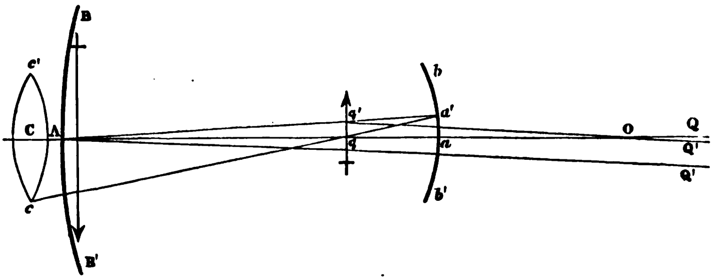
and the magnifying power $= l\left(\frac{1}{f'} - \frac{1}{l}\right) \times \frac{f_o}{f_e}$

$$= \frac{(l-f')f_o}{f'f_e}$$

$$= \frac{f_o^2}{f'f_e} \text{ nearly}$$

We see that with the same object-speculum and eye-piece the magnifying power is greater as f' is less; and this telescope is generally furnished with two or more small mirrors, of different focal lengths, fixed to upright arms, which can be slided readily in and out of the tube of the instrument, to change the magnifying power, as well as different eye-pieces of different powers for the same purpose.

Art. 92. Prop. *To find the angular magnitude of the field of view in Gregory's telescope.*



Let BAB' be the object-mirror, supposed continuous, bab' the small mirror, and cCc' the eye-glass, of which the half-aperture $Cc = a$.

We suppose the small mirror sufficiently large to reflect the pencils from the large mirror completely, and therefore the field of view will depend on the aperture of the eye-lens.

Let q' be the point in the inverted image formed by the object-mirror, where it is cut by a ray Aa' reflected at A, and reflected again by the small mirror at a', and finally touches the edge of the lens c; then q' is the point which is seen by half-pencils, and may be taken as the boundary of the field of view; and the angle $q'Oq$, which qq' subtends at the center of the curvature of the mirror, is the angular radius of the field of view required.

Since q is the bisection of AO, the angle $q'Oq$ equals the angle $q'Aq$. The line $a'c$ will pass very nearly through q; for the second image being formed very near A, these points A and q will be conjugate foci to the small mirror very nearly.

Now $$\text{angle } a'Aa = \angle q'Aq = \frac{aa'}{Aa} \text{ nearly,}$$

and $$\text{angle } a'qa = \angle cqC = \frac{aa'}{qa} \text{ nearly,}$$
$$= \frac{Cc}{Cq}$$

therefore substituting the value of $aa' = \dfrac{Cc}{Cq} \cdot qa$

$$\text{angle } q'Oq = \frac{Cc}{Cq} \cdot \frac{qa}{Aa}$$

and using the same notation as in the last Article, and putting f' for aq approximately, we have

$$\text{The angular radius of the field of view} = \frac{Cc}{Cq} \cdot \frac{qa}{Aa}$$
$$= \frac{a}{f_o + f_e} \times \frac{f'}{f_o + f'}$$
$$= \frac{a \cdot f'}{f_o^2} \text{ as an approximation,}$$

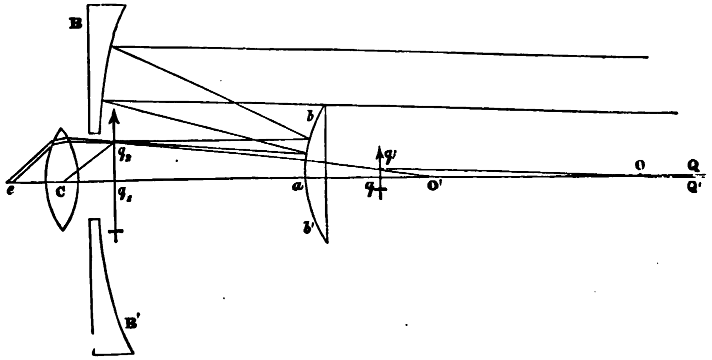
since f_e and f' are small compared with f_o.

ON CASSEGRAIN'S TELESCOPE.

Cassegrain's telescope differs from Gregory's only in having in place of the concave small mirror a convex one, which is placed nearer the large mirror than its focus, instead of beyond it. The pencils from the large mirror therefore, fall in a state of convergence upon the small convex one; and after reflexion by it will form a real inverted image, as at q_1q_2 in the figure, by the second part. of Art. 22.

Art. 93. Prop. *To find the magnifying power of Cassegrain's telescope.*

Let BB' be the large speculum, bab' the small convex one, of which the centers are O and O' respectively, and C the eye-glass. Let qq' be the inverted image of a distant object, which would have been formed in the principal focus of the large mirror, if the pencils had not been intercepted by the small one; and q_1q_2 the real image of the virtual one qq', formed by the small convex mirror in the focus of the eye-glass C; q_2 and q' being points in the images corresponding to a point Q' in the object, and q_1, q those corresponding to a point Q.



Join q_2 and C, then the angle q_2Cq_1 is the angle under which the points q_2 and q_1 are seen by an eye at e; and the angle $q'Oq = Q'OQ$ is that under which the corresponding points in the object are seen: hence, using the same notation as before, the magnifying power $= \dfrac{\angle q_2Cq_1}{\angle q'Oq}$

$$= \frac{\dfrac{q_2q_1}{Cq_1}}{\dfrac{q'q}{Oq}}$$

$$= \frac{q_1O'}{qO'} \cdot \frac{f_o}{f_e} \text{ by the second part of Art. 22}$$

$$= \frac{q_1a}{qa} \cdot \frac{f_o}{f_e} \text{ by Art. 20}$$

Let the distance between the two specula $= l = aq_1$ nearly. From Art. 20

$$\frac{1}{aq} = \frac{1}{f'} + \frac{1}{q_1a}$$

$$= \frac{1}{f'} + \frac{1}{l}$$

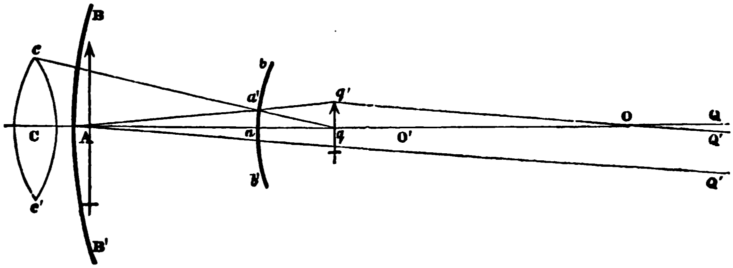
therefore the magnifying power $= l\left(\dfrac{1}{f'} + \dfrac{1}{l}\right)\dfrac{f_o}{f_e}$

$$= \frac{l+f'}{f'} \cdot \frac{f_o}{f_e}$$

$$= \frac{f_o^2}{f'f_e} \text{ nearly.}$$

We see that the expression for the magnifying power in Cassegrain's is the same as in Gregory's telescope, and it has the advantage of being shorter; but for day purposes it is inapplicable, as the image is inverted.

Art. 94. Prop. *To find the angular magnitude of the field of view in Cassegrain's telescope.*



Let BAB' be the object-mirror supposed without perforation in its center, O the center of its curvature, bab' the small mirror, and $c'Cc$ the eye-lens. Let f_o, f', f_e and a be as before. Let Q be the point in the object to which the axis CAO of the telescope is directed, Q' the point from which the ray reflected at A, and afterwards at a' on the small mirror, passes in $a'c$ to the edge of the eye-glass; if we produce Aa' to cut the virtual image in q', then Q', the corresponding point in the object to q', will be seen by half-pencils, and may be considered as the limit of the field of view. Drawing the line $q'OQ'$, the angle $Q'OQ = \angle q'Oq$ is the angular field of view to be found.

The line ca' when produced will cut the axis nearly in q, because q and A are conjugate foci, nearly, of the small mirror. Now angle $q'Oq = \angle q'Aq$, because q bisects AO,

and
$$\angle q'Aq = \angle a'Aa = \frac{aa'}{Aa}$$
$$= \frac{aa'}{f_o - f'} \text{ nearly}$$

also
$$\angle cqA = a'qa = \frac{aa'}{aq}$$
$$= \frac{cC}{qC}$$

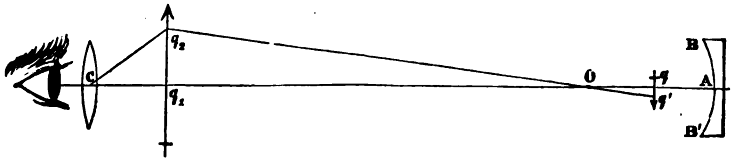
whence
$$aa' = \frac{cC}{qC} \cdot aq$$
$$= \frac{a}{f_o + f_e} \cdot f'$$

Then the angular radius of the field of view $= \angle q'Aq$
$$= \frac{a \, . f'}{(f_o - f')(f_o + f_e)}$$
$$= \frac{a \, . f'}{f_o^2} \text{ approximately.}$$

ON SIR ISAAC NEWTON'S REFLECTING MICROSCOPE.

This is the simplest of the compound microscopes, the essential parts of the instrument being a concave object-speculum and an eye-lens. The object being placed between the center of the curvature and the principal focus of the speculum, a magnified real image is formed in the conjugate focus; and this is again magnified on being seen through the eye-glass.

Let O be the center of curvature of the speculum BAB', and C the eye-glass.



If qq' be a small object, and we draw lines through the points q' and q and the center O, the conjugate foci of the reflected pencils will be in these lines, as at q_2 and q_1 respectively; and an image will be formed, of which the magnitude is to that of the object as Oq_1 to Oq, or by Art. 19, as Aq_1 to Aq.

If Cq_1 be such a distance that the pencils, after emerging from the eye-glass, diverge from a virtual image at the least distance of distinct vision, as in Art. 89, the magnifying power is the number of times the angle which the image subtends at the eye contains the angle which the object itself subtends at the least distance of distinct vision. It is also the product of the magnifying powers of the eye-lens and the mirror,

$$\text{or the magnifying power} = \left(\frac{e-d+f}{f}\right)\frac{Aq_1}{Aq}$$

where e, d, and f are as in Art. 89. Let the distance of q_1 from $A = l$, which is the length of the tube of the microscope minus the focal length of the eye-lens nearly, and f_0 the focal length of the mirror, then

$$\text{the magnifying power} = \left(\frac{e-d+f}{f}\right)l\left(\frac{1}{f_0}-\frac{1}{l}\right)$$

$$= \frac{(e-d+f)\,(l-f_0)}{ff_0}$$

When transparent objects are to be viewed by the light passing through them, a small plane mirror is required to be placed in the axis of the tube at an angle of 45°. In Amici's modification this plane mirror is placed so between *A* and *q* that the object may be placed outside the tube. The objection to this is, that the errors of workmanship in the plane mirror will diminish the effect of the instrument. A plan used by the author was to place the transparent objects at *q* on small bits of mica, cemented to pins, with small wooden handles, which could readily be placed in the adjoining part of the microscope, and to throw the light from a lens at the side of the tube, through them by a small plane mirror *behind them*, which was set at 45° to the axis of the tube. In this method the image formed by the concave mirror was direct, as in the figure.

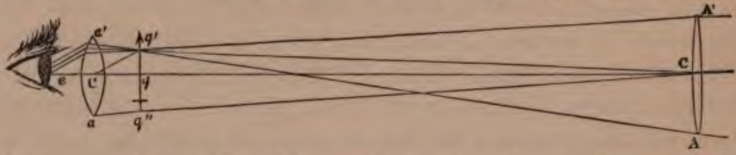
Though very good effects have been obtained by this microscope, there is room to expect much greater, when as great skill and attention shall have been paid to it as have latterly been expended on the achromatic compound microscope. The advantages of there being only one surface to polish and bring to a correct figure in the concave mirror, and reflexion being without any separation of colours, are very great, when we recollect that an achromatic lens is only an approximately corrected instrument.

In comparing the finest reflecting instruments with the finest refracting ones, a sharp distinctness of the outlines is shewn by the former; whilst, however accurate the images shewn by the latter, they have a haziness on all their outlines.

For the proper mode of using microscopes and testing their relative powers, the student may consult with advantage Sir David Brewster's treatise on the microscope.

ON THE ASTRONOMICAL TELESCOPE.

The astronomical telescope in its simplest form consists of two convex lenses, as in the figure, where *ACA'* represents the object-glass of a long focal length, and *aca'* the eye-glass, of which the focal length is short compared with that of the object-glass. An inverted image *q'qq''* of a distant object is formed at the common focus of the object and eye-glasses, whose distance is therefore the sum of their focal lengths. A pencil diverging from any point in the object, which is brought to a focus at q' in the image, will fall upon the eye-glass in such a state of divergence that it will emerge from it in a state of parallelism, and therefore be fit for vision to an eye at *e*.



The eye-tube of the telescope, or that which carries the eye-glass, has an eye-hole near *e*, where every ray passing through *C*, the center of the object-glass, is brought to cross the axis of the telescope; and *e* is the conjugate focus to *C* for the eye-glass; or *ec* is rather more than the focal length of the lens *aca'*. If the eye be placed at *e*, all the pencils of parallel rays refracted by the eye-glass will enter it, and the full field of view will be visible at once; but if the eye be placed either nearer or farther away, some of the extreme pencils will not enter the pupil, and only the central parts, more or less according to the distance of the eye from *e*, will be seen.

ART. 95. PROP. *To find the magnifying power of the astronomical telescope.*

If q be the image of the point in the object to which the axis of the telescope is directed, and q' the image of another point, then $q'Cq$ is the angle under which those points are seen at the object-glass. If we draw the line $q'c$ through the center of the eye-glass, this ray will pass through it without deviation, and gives us the direction of the parallel emergent rays; so that the angle under which the points q and q' in the image are seen by the eye is the angle $q'cq$.

$$\text{Therefore the magnifying power} = \frac{\angle q'cq}{\angle q'Cq}$$

$$= \frac{\dfrac{q'q}{cq}}{\dfrac{q'q}{Cq}} \text{ nearly}$$

$$= \frac{Cq}{cq}$$

$$= \frac{f_o}{f_e} \text{ if } f_o \text{ be the focal length of}$$

the object-glass, and f_e that of the eye-glass.

Art. 96. Prop. *To find the field of view of the astronomical telescope.*

If we draw the line Ca through the center of the object-glass to the edge of the eye-glass, cutting the image in q'', we see that of the rays of the pencil, which are brought to a focus at q'', one half will fall on the eye-glass, and be by it refracted to the eye, but the other half will pass outside the edge a. The point q'', being seen by half-pencils, may be taken as the boundary of the field of view, of which the angular measure is the angle $q''Cq$; or

$$\text{the angular radius of the field of view} = \angle q''Cq$$
$$= \angle aCc$$
$$= \frac{a}{f_o + f_e}, \text{ if } a \text{ be half the}$$

aperture of the eye-glass.

If we joined A' and a, the point where this line would cut the image would be the limit of the parts seen by full pencils, or the brightest part of the field of view; and if we joined A and a, the point of the image cut by that line would be the extreme part from which a ray would reach the eye-glass, or the extreme of the field of view. The part between these two points is the *ragged edge*, which may be cut off to a greater or less extent by a diaphragm at the image, with a circular aperture of the desired radius in its center.

We saw in Art. 95 that the magnifying power might be increased indefinitely by using an eye-glass of shorter and shorter focus; but in practice there are several causes which set a limit to the magnifying power that can be used with a given object-glass. If the object-glass be a single convex lens, the chromatic dispersion renders the image indistinct, unless the aperture be small compared with the focal length; but the brightness of the image being proportional to the quantity of light on each point of it, or proportional to the area of the object-glass, it becomes faint when the aperture is small, and will only bear a low magnifying power. The magnifying power is increased by increasing the focal length of the object-glass, which allows the aperture to be increased at the same time: thus the long telescope of Huygens, called *aërial* from being without a tube, had an aperture of six inches, and a focal length of one hundred and twenty-three feet, bearing a magnifying power of two hundred and eighteen times.

The discovery of the achromatic lens formed a new era in the history of the refracting telescope; and the achromatic and aplanatic object-glasses, with Huygenian eye-pieces, of the astronomical telescopes now used, are of apertures which would have required focal lengths ten or twelve times as great in the old telescopes.

The magnifying power which a telescope will bear is different for the different heavenly bodies, the moon and planets not allowing so high a power as the fixed stars; and on the other hand, the brightness of the sun is so great, that a telescope sufficient to shew the nature of his surface around the spots which are frequently visible, may be made with a spectacle-lens of three or four feet focus, with an eye-lens of one inch or less focus, and a plane darkening glass before the eye which transmits only light of one colour.

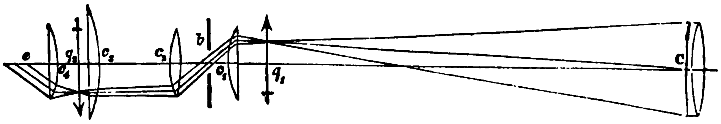
The undulating state of the atmosphere, from the continual intermingling of portions of air of different temperatures, and therefore of different refractive powers, is a great impediment to the correct action of telescopes, and to their bearing high magnifying powers. Those who wish to learn the method of using the astronomical telescope, may consult with advantage

Dr. Kitchiner's *Economy of the Eyes*, part ii. *on Telescopes, being the result of Thirty Years' Experiments with Fifty-one Telescopes.*

ART. 97. *To explain the construction of the achromatic day-telescope.*

The astronomical telescope shews inverted images of the objects towards which it is turned, and for astronomical objects this is no disadvantage; but in a telescope to be used by day upon terrestrial objects, such an inversion is not admissible. The inversion may be remedied, as in the Gregorian telescope, by forming an image of the first image, which may be done by the addition of another convex lens; but to secure a large and accurate field of view, the usual *terrestrial eye-tube* contains four convex lenses, as in the next figure.

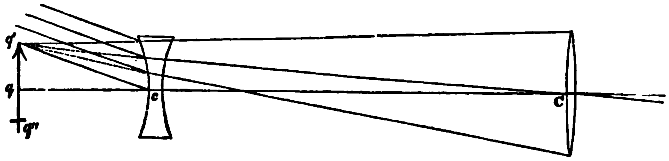
The inverted image of a distant object being formed by the object-glass C at q_1, the pencils diverging again fall on the lens c_1, which is nearly its focal length distant from q_1; the refracted pencils then all cross the axis of the telescope at a point b, where a stop or diaphragm is placed to cut off all stray light; falling then on a convex lens c_2, they are bent towards the axis, and after falling on the field-glass c_3, they are brought to a focus again, forming a second image, which is erect at q_2; this image being in the focus of the eye-glass c_4, the pencils refracted by it on emergence consist each of parallel rays, and are therefore fit for vision to an eye at e.



ON THE GALILEAN TELESCOPE.

The telescope of Galileo consists of a convex object-lens and a concave eye-lens, set at a distance equal to the difference of their focal lengths. This construction has the advantage of shewing objects erect; it is shorter than the other forms; and when made with single lenses, the errors of the concave correct in some degree those of the object-lens: for these reasons it is used for opera-glasses and perspective telescopes of low power; but it has the disadvantage of having a very small field of view when the magnifying power is considerable.

Art. 98. Prop. *To find the magnifying power of Galileo's telescope.*



Let C be the object-glass, c the eye-glass, f_o and f_e their focal lengths respectively, and let their distance Cc equal $f_o - f_e$. The image $q'qq''$ of a distant object would be formed at the principal focus of the object-glass if the eye-glass did not intercept the pencils; but the image being also at its principal focus, each pencil converging to a point in the image at incidence, will emerge from it in a state of parallelism, and therefore fit for vision to an eye close behind it.

If q be the point in the image which is in the axis of the telescope, q' any other point, the corresponding points in the object will subtend at the object-glass the angle $q'Cq$. If we draw the line $q'c$, this would be the direction of a ray passing through the eye-glass without deviation; and therefore the direction of the parallel rays, which before incidence on the eye-glass were converging to q', and hence the angle $q'cq$ is that which the virtual image subtends to the eye.

$$\text{The magnifying power} = \frac{\angle q'cq}{\angle q'Cq}$$

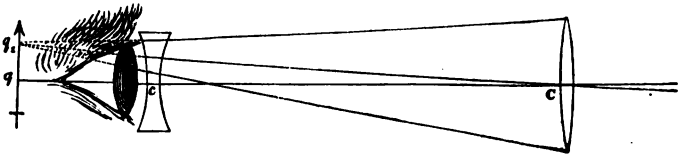
$$= \frac{\dfrac{q'q}{cq}}{\dfrac{q'q}{Cq}} \text{ nearly}$$

$$= \frac{Cq}{cq}$$

$$= \frac{f_o}{f_e}$$

We see that the rays from various points in the object enter the eye in the same relative directions as they enter the naked eye, and therefore the virtual magnified image is seen erect.

Art. 99. *To find the field of view in the Galilean telescope.*



The eye being placed close to the eye-glass, as in the figure, let q_1 be the point in the image determined by drawing a line through C the center of the object-glass, and the upper edge of the pupil of the eye; then the point in the object corresponding to q_1 will be seen by half-pencils, and may be taken as the limit of the field of view. Let the half aperture of the pupil equal a, then

$$\text{the angular field of view} = \angle q_1 C q$$

$$= \frac{q_1 q}{qC}$$

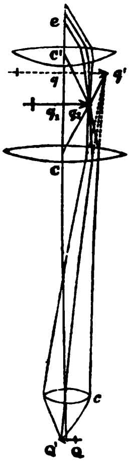
$$= \frac{a}{cC}$$

$$= \frac{a}{f_o - f_e}$$

This value is always small, and becomes smaller as f_o is increased and f_e diminished, or as the magnifying power is increased.

ON THE COMPOUND REFRACTING MICROSCOPE.

In the ordinary compound microscope the object-lens is a double convex of short focus, as c in the figure. A small object being placed at QQ', an image of it would be formed in the conjugate focus qq'; but the field-lens C being interposed, the pencils are brought to a focus at q_1q_2, and the magnified image formed there is viewed by means of an eye-glass c' by an eye situated at e.



Many variations of the annexed form have been constructed; as for instance, two lenses near together are sometimes substituted for the one c', and the field-glass removed farther from them; and in general the aperture of the object-glass c is limited, to produce a more distinct though less bright image. A diaphragm is also frequently placed between the object-glass and field-glass, near to the former, the circular aperture in which allows a certain magnitude to the pencils forming the central parts of the field, and causes the outer parts to be formed by pencils passing excentrically through the object-glass, so as to make them to be seen more accurately than they otherwise would be.

ART. 100. *To find the magnifying power of the compound refracting microscope.*

In finding the magnifying power of telescopes, we supposed the visual pencil on emerging from the eye-glass to consist of parallel rays, as that entering the naked eye would, when coming from a distant object. In finding the magnifying power of a microscope, we have to compare the angle the object subtends to the naked eye at the *least distance of distinct vision*, with the angle which the image formed by the microscope subtends when the visual pencil enters the eye in the same state of divergence, or as if coming from a virtual image at the least distance of distinct vision.

Let f be the focal length of the object-glass c, f' that of the field-glass C, f_e that of the eye-glass c'. Let $cq = l =$ the length of the instrument nearly, $e =$ the least distance of distinct vision.

$$\text{The magnifying power} = \frac{\text{the angle } q_1q_2 \text{ subtends at the eye}}{\text{the angle } QQ' \text{ subtends at the distance } e}$$

$$= \frac{\dfrac{q_1q_2}{c'q_1}}{\dfrac{QQ'}{e}}$$

$$= \frac{e}{c'q_1} \cdot \frac{q_1q_2}{QQ'}$$

$$= \frac{e}{c'q_1} \cdot \frac{q_1q_2}{qq'} \cdot \frac{qq'}{QQ'}$$

$$= \frac{e}{c'q_1} \cdot \frac{Cq_1}{Cq} \cdot \frac{cq}{cQ}$$

From Art. 57 we have

$$\frac{1}{cq} = \frac{1}{f} - \frac{1}{cQ}$$

$$-\frac{1}{Cq} = \frac{1}{f'} - \frac{1}{Cq_1}$$

$$-\frac{1}{e} = \frac{1}{f_e} - \frac{1}{c'q_1}$$

Substituting from these in the preceding expression, we have

$$\text{the magnifying power} = \left(\frac{e}{f_e} + 1\right)\left(1 - \frac{Cq_1}{f'}\right)\left(\frac{l}{f} - 1\right)$$

If the effect of the field-glass is inconsiderable, so that $\frac{Cq_1}{f'}$ is small, at the same time that $\frac{e}{f_e}$ and $\frac{l}{f}$ are large,

$$\text{the magnifying power} = \frac{e}{f_e} \cdot \frac{l}{f} \text{ nearly.}$$

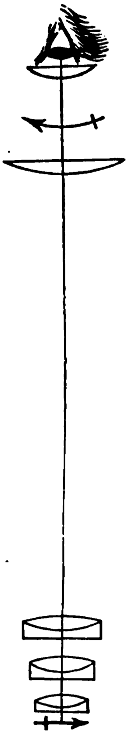
As the focal lengths of the object-glasses of a microscope, or as they are generally termed, *the powers*, cannot be determined very accurately in a direct manner, it is necessary to measure the magnifying power experimentally when it is required to be known to considerable nicety. This is performed by what is called *the method of false vision*, which may be very conveniently used also to find the magnifying powers of telescopes. A small object of known magnitude being placed before the object-glass of the microscope, its image is viewed by one eye through the instrument, whilst the other eye is directed to a divided scale at the least distance of distinct vision; when the microscope is adjusted, the scale and the image of the small object will be seen together, and the magnitude of the image can be read upon the scale. This magnitude divided by the real magnitude of the object gives the magnifying power required.

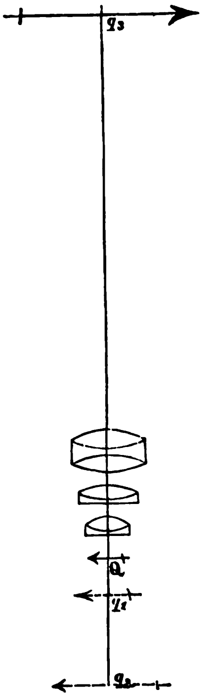
When the magnifying power of a telescope is not very great, it may be found by turning it towards a brick wall not more distant than that the courses of bricks can be counted by the naked eye: the number of courses to the naked eye which correspond to one course seen at the same time by the other eye through the telescope gives the magnifying power for that distance of the object.

A great improvement was effected in the compound refracting microscope by the introduction of achromatic lenses of short foci for the powers.

For some years past the achromatic microscope has received great attention, and has been brought by the artists to a high degree of perfection in workmanship. Achromatic object-glasses or powers having been introduced, Mr. Lister found that by using, for his powers, combinations of two or more, separated by determined intervals, much larger pencils could be brought to accurate foci, and the instrument would bear with distinctness much higher magnifying powers than with a single achromatic object-glass. Mr. Ross, the eminent practical optician, also discovered that the distances between the achromatic lenses required to be changed for the most distinct effect, when the microscope which had been adjusted for viewing objects directly, was applied to those which were preserved in Canada balsam or otherwise, under thin plates of glass.

One of the figures annexed represents the achromatic microscope as formerly constructed, with an object-lens, consisting of three plano-convex achromatic lenses, and a Huygenian eye-piece. The other figure represents a later form of the compound object-glass, with the positions of the successive foci; but further improvements have been made in the construction and will continue to be made. In the fitting up of the best triple object-lenses there is a means of changing the distance of the lower lens from the two upper ones in accordance with Mr. Ross's improvement.





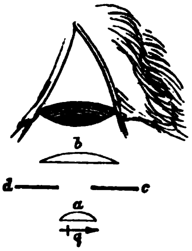
It was considered a great achievement when an achromatic object-lens was made which would bring a pencil of an angle of near twenty degrees to an accurate focus; but since Mr. Lister's views have been carried out, triple lenses have been made, which bring pencils accurately to a focus, of angles of more than ninety degrees.

The students who wish to learn more particulars on this subject may consult the article 'Microscope' in the *Penny Cyclopædia,* written by Mr. Ross.

ON DR. WOLLASTON'S DOUBLET.

Dr. Wollaston conceived the idea that Huygens's eye-piece turned the reverse way to the eye would form an excellent microscope, which is found to be the case; although the principles on which its advantages as an eye-piece depend do not hold in the same way in this use of it, as shown at page 184, PART II. The practical opticians have improved the construction by introducing a diaphragm between the lenses, so that a large angular field is procured sufficiently accurate to show many difficult, or as they are called, *test objects.*

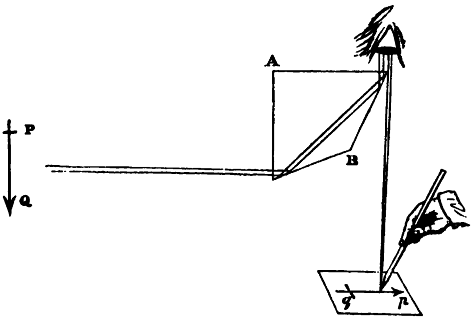
In the annexed figure, *q* shews the place of the object; *a* and *b* are the plano-convex lenses, the focal length of *b* is three times that of *a*, but they are drawn very large compared with the actual magnitudes and focal lengths which are frequently used; *dc* is the diaphragm.



The doublet must be classed with the single microscopes, and its magnifying power is determined by the results of Article 89, after the focal length of the equivalent single lens has been found, by Art. 81.

ON DR. WOLLASTON'S CAMERA LUCIDA.

This instrument consists of a small quadrilateral prism of glass, of which *AB* in the figure is the perpendicular section, held in a brass frame which is attached to an upright rod having at its lower end a screw clamp to fix it to the edge of a table. The prism being at the height of about a foot from the table, has its edges horizontal. Two of its faces, as in the figure, are at a right angle at *A*, and the faces contiguous make respectively with them angles of 67½ degrees; so that the remaining obtuse angle at *B* contains 135 degrees.

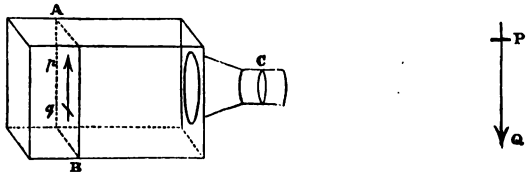


Rays coming from an object *PQ* and falling nearly perpendicularly on the first surface, enter the prism and undergo total reflexion at the contiguous surface; they then fall at the same angle on the next surface, and are totally reflected again; and finally emerge nearly perpendicularly to the remaining surface. An eye, as in the figure, receives the emergent pencil through one part of the pupil, so that an image, *pq*, of the object is seen projected upon a sheet of paper upon the table. The rays from the drawing-pencil passing the edge of the prism enter the other part of the pupil, so that the pencil and image are seen together upon the paper, and a sketch of the latter can be taken. Strictly, the image is equally distant from the prism with the object, and so cannot be seen distinctly at the same time with the drawing-pencil; but a plate of metal, with a small aperture as an eye-hole, being placed at the edge under the eye, so that the rays through the prism and those from the drawing-pencil which both pass through the eye-hole form only very small pencils, the difficulty is greatly diminished, and many artists have acquired great readiness in using the instrument. Others, however, have never been able to acquire this facility; and other modifications of the same principle have been proposed to remedy the inconvenience which they have felt.

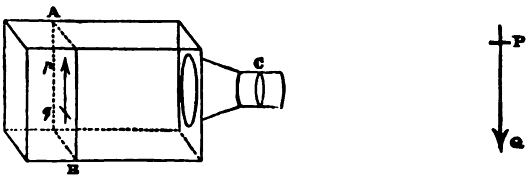
The reasons for two reflexions being used, instead of one from a single metallic mirror, will be seen from the discussion in Article 8.

ON THE CAMERA OBSCURA.

The camera obscura is constructed in a variety of forms; but in all, the intention is to receive upon a screen, or other surface, the real images formed by a convex lens of external objects. Amongst the uses to which it is applied, none is of greater interest than photography, by which a permanent picture of the image formed in the camera is procured. The form of the instrument as used for this purpose is similar to the figure, which represents a rectangular box of wood, with a sliding tube at its fore part containing, as at *C*, a convex lens: at *AB* a screen of thin paper, or a



sheet of glass ground rough on one surface, is placed in a fixed position in the inside of the box; and the back of the box is either left open, or has a door to open or shut.



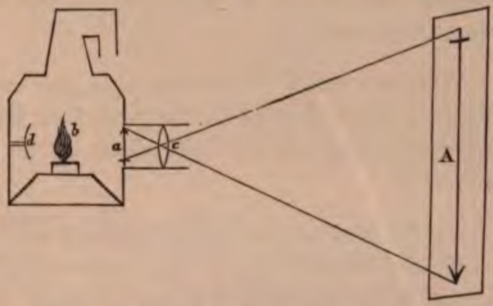
The instrument being turned towards an object *PQ*, and the lens being set, by means of the sliding tube, to the proper distance from the screen *AB*, there will be formed upon it an inverted image *pq*, of the object, which will be seen through the open back of the box, upon the translucent screen. When the camera is applied to photography, the lens should be adjusted to give the most distinct image with the violet rays, as the chemically acting rays accompany that end of the solar spectrum. The screen *AB* being removed, and a prepared sensitive surface on glass, paper, or silvered copper-plate being put in the same place, and having been preserved in close covers from the action of light until the required time, the image is allowed to fall upon it; and after a short time, this image produces an effect upon the sensitive surface from which subsequent chemical processes develope the photographic picture of the object.

The inversion of the picture as to the up-and-down directions, is taken away when it is turned the reverse way to what it is in the instrument; but there remains the inversion of right and left. This latter imperfection is avoided in many of the methods of photography.

In those forms of the camera obscura where a large picture is to be received upon the screen, this latter must be curved to the same curvature as the image, so that the foci for the oblique pencils may fall upon it.

ON THE MAGIC LANTERN.

The use of the magic lantern is to throw upon a screen in one focus of a convex lens a magnified image of a picture painted upon glass in the other focus, which is strongly illuminated by the light from the lamp within the lantern. The simplest form is that given in the figure, where *a* is the position of the picture, *b* the flame of the lamp, and *c* the convex lens, of which *A* is the focus conjugate to *a*, and therefore the position of the screen on which the image is received. The picture being inverted at *a*, the image at *A* is erect. The lens is fixed in a sliding tube, so that it may be brought to the proper distance from *a*, for forming a correct image upon the screen. As the image is greatly magnified, and has over its whole area the same quantity of light as the picture, minus the loss in passing through the lens, therefore, although viewed in a dark room with the light within the lantern carefully prevented from shining out of it otherwise than through the lens, yet, in order that it may be seen brightly, the light passing through the picture must be very strong. To increase the brightness of the image, the magic lantern has generally a concave spherical reflector, as at *d* in the figure, or one or more short-focused lenses between *b* and *a*, to condense a strong light upon the picture, or both of these means are used at the same time.



The oxyhydrogen microscope is only a particular form of the magic lantern; the brilliant light from a cylinder of lime acted on by the oxyhydrogen blowpipe being substituted for the flame of a lamp, and several condensing lenses placed between it and the picture. To procure a more correct image, it is usual to put two lenses in place of the single lens *c*; and by having a means of changing the distance between them, and at the same time keeping the picture and screen in the conjugate foci, we can change the magnifying power, as shown in Art. 81, and thus produce the effect of the image appearing to approach the spectators as it is enlarged in magnitude, and to recede from them as it is diminished. This is generally produced by the phantasmagoria lantern; the spectators viewing the image through a translucent screen, whilst the lantern on the opposite side is slided along a surface nearer to or further from the screen, and the image kept distinct, though of varying magnitude, by adjusting the lens to the proper distance from the picture.

What are called dissolving views, are produced by having two lanterns directed towards the screen at the same time, and gradually shutting out the light of one lantern as that of the other is admitted.

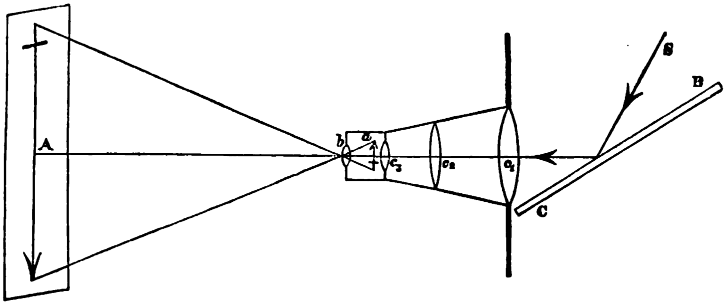
The lucernal microscope is the magic lantern adapted to shewing images of natural objects upon a screen, such as the transparent wings of insects, thin sections of wood, &c. The modification required is, that the condensing lenses must have sufficient power to collect the illuminating light into the space allotted for the object.

In these different forms of the magic lantern the edges of the field over which the image is spread will never be very distinct at the same time with the center, unless the screen has the same curvature as the image; or otherwise the picture, &c., is upon such a curved surface as to give a flat image.

ON THE SOLAR MICROSCOPE.

The solar microscope is the same in principle with the lucernal microscope and magic lantern, the difference being that the light of the sun is used to illuminate the object whose image is to be thrown on the screen. It is generally used to show magnified views of natural objects; but is also applicable, with similar lenses, to shew the images of pictures, like the oxyhydrogen microscope and magic lantern.

The figure represents the optical construction of the instrument, which is fastened to a window-shutter of a darkened room, at an aperture in it, which is a little larger than will admit a beam of the sun's light of the diameter of the first condensing lens c_1.



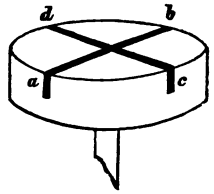
The body of the microscope, which contains the condensing lenses c_1, c_2, c_3, and the power *b*, is horizontal, and the sun's light is reflected through it by the mirror, of ordinary looking-glass, *BC*. The condensing lenses collecting an exceedingly strong light upon a small object at *a* in one focus of the power *b*, a highly magnified and very bright image is received upon a screen at *A* in the conjugate focus.

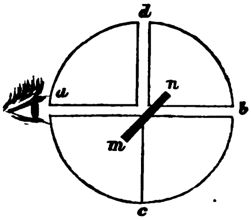
The solar microscope, the lucernal and oxyhydrogen microscopes, as used to shew magnified images of natural objects, are adapted to exhibit them in an agreeable manner to a company of spectators at the same time, rather than to shew their minute structure, as Wollaston's doublet and the achromatic microscope will shew them.

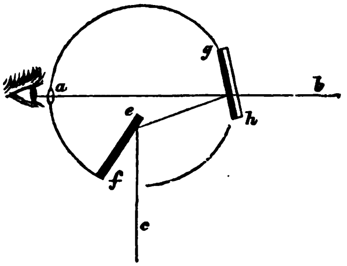
ON THE OPTICAL SQUARE.

The optical square is an application of the property of the reflection at two plane mirrors, by which a much more convenient and expeditious instrument is produced to supply the place of the cross used by surveyors to determine directions at right angles to each other.

The result to be obtained will be more easily understood by examining the construction of the surveyor's cross-staff, as shewn in the upper figure, where *ab* and *cd* are grooves, at right angles to each other, sawn into the circular head of the staff. The staff being stuck into the ground, the eye, on looking along the grooves in succession, sees the objects whose directions from the center of the cross are at right angles to each other. The inconvenience and delay of first looking along one of the grooves and then along the other might be remedied by placing a mirror, as *mn* in the second figure, at an angle of 45° with the direction of *ab*, and sunk so far in the head of the staff that the eye looking over the mirror would see in the direction *ab*, and looking into the mirror would see in the perpendicular direction *cd*.



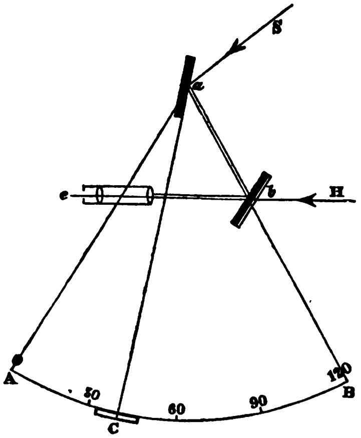




The optical square is, however, a better instrument than either of these. It was shewn in Art. 11, that the deviation of a ray of light reflected by two plane mirrors in a plane perpendicular to their intersection equals twice the angle between the mirrors; so that if the angle between the mirrors is 45°, the deviation is a right angle. If *ef* and *gh* are two such mirrors of silvered plate-glass, and the silvering of *gh* be removed from its outer half, the objects beyond will be seen through this part at the same time with a faint reflection of the light coming from the other mirror at the surface of the glass. The object in the direction of *ab* will seem coinciding with the image of an object in a direction *ec*, at right angles to *ab*, given by the two reflexions; and thus the perpendicular to a given direction is determined with accuracy and expedition.

ON HADLEY'S SEXTANT.

Hadley's sextant is an instrument for measuring the angle between two given directions, by means of the property of two plane mirrors inclined to each other, which is investigated in Article 11. The mirrors, which are of silvered plate-glass, are placed as at *a* and *b* in the figure, and both of them perpendicular to the plane of the instrument; the one at *b* is fixed to the frame of it, and has the silvering removed from its outer half; the other at *a* is fixed to an arm turning about a pivot at the center of the graduated arc *AB*.



The arm to which the mirror *a* is fixed carries a vernier at *C*; and when the index on the vernier points to 0 on the graduated arc, the mirrors are parallel. An eye-hole at *e*, or a small telescope directed to the mirror *b*, is fixed to the frame.

Rays of light passing through the unsilvered part of the mirror *b*, from a distant object, as the offing at sea, and others reflected by it parallel to them, will be brought to a focus together in the telescope; and if these latter are rays from a star or other heavenly body which have been reflected by the mirror *a* to *b*, the image of the star, &c. will coincide with that of the offing. The deviation of the ray *Sa* from the star is its altitude above the offing, and is, by Article 11, twice the angle between the mirrors, or twice the degrees contained in the arc *AC*; so that half degrees on the graduated arc being marked as whole degrees, the reading by the vernier gives the altitude of the star, &c.

By the same method the angle subtended by any two objects can be measured.

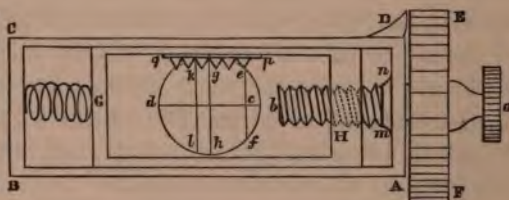
On land the altitudes of the heavenly bodies are determined with this instrument by measuring in a similar manner the angle between the direct rays from the star and those from its image in the horizontal surface of mercury or other fluid, which is then called an *artificial horizon.* See the proof of the law of reflexion in the Introductory Chapter.

ON THE REFLECTING CIRCLE.

The reflecting circle is constructed on the same optical principle as Hadley's sextant; but the whole circle is graduated, and there are two or three verniers at equal distances, so that the angles are read off at different parts of the graduated circle; and the average gives a much more correct reading than can be expected from the sextant. It is used in the same cases as the sextant, where stationary instruments are not applicable, and the objects are supposed to be very distant, so that the space between the mirrors subtends no appreciable angle to them. When the objects are near, an error is introduced which is called *the parallax* of the instrument.

ON THE MICROMETER.

The micrometers, as their name implies, are used to determine the measures of small magnitudes. There are several forms; but the only one which will be described here is the screw micrometer.



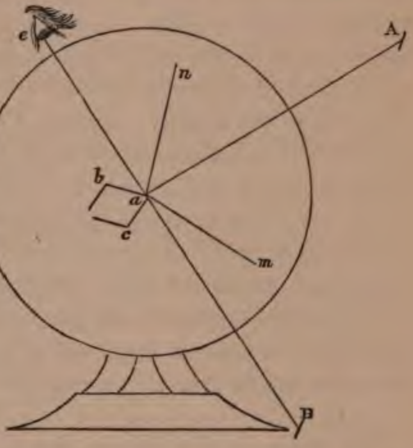
A rectangular brass frame *ABCD* has its faces parallel to the plane of the paper closed on each side by a brass plate, except a circular aperture through each, where are seen the longitudinal spider's line *cd*, and the perpendicular ones *ef*, *gh*, *kl*; *cd*, *ef*, *gh* are fixed in the instrument, but *kl* is attached to the moveable rectangular frame *GH*, which is screwed backwards and forwards by the screw of which the axis is *ab*, and it passes closely to the other lines without touching them. A spiral spring, acting as in the figure at *G*, keeps the moveable frame pressing in one direction against the threads of the screw, and the shoulder *mn*, upon it against the end *AD*, of the frame. A circle *EF*, is fixed on the axle of the screw where it has passed through the end of the frame, and its circumference is divided into 50 or 100 parts. On the screw being turned round by means of the milled head near *a*, the parts of a revolution are read off by means of the index fixed near *D*. The number of revolutions of the screw which the moveable line *kl*, is distant from the central one *gh*, is given by the teeth of a rack at *pq*. The distance of the lines *ef* and *gh* being known, greater measures can be taken than could be with *gh* only.

The whole set of lines are placed in the common focus of the object-lenses of the telescope or microscope and the Ramsden's eye-piece, so that they appear to coincide with the image which is there formed, and consequently, being seen with it, they afford the means of determining its magnitude with very great accuracy. The object and image subtend the same angle at the center of the object-glass; and when the distance of the image from that point and its magnitude have been measured, the angle is known. If the distance of the object is also known, a simple calculation gives its magnitude. There are several modifications in the construction, which are adapted for particular cases of the application of the micrometer.

ON DR. WOLLASTON'S GONIOMETER.

This is an instrument for measuring the angles between the faces of crystals, by means of the directions of a beam of light reflected by them.

A graduated circle has its axis projecting on each side, and is supported by the axis on one side passing through a circular hole in the upright frame-work fixed to the base. The axis on the other side carries three arms connected together, which allow motion in three rectangular directions. One or more verniers are fixed to the frame-work, and serve to determine the angle through which the circle is turned.



The crystal is attached with wax to a plate on the last of the arms, and is, by means of them, placed in such a position that the intersection of two of its faces is at the axis, and perpendicular to the plane of the circle, with which it may then be turned round. Let *ab*, *ac* be sections of two faces of a crystal meeting at *a* in an edge which is at the axis of the circle and perpendicular to its plane. Let *an*, *am* be normals to the faces at *a*; the angle *man* is the supplement of the angle *bac*, which will therefore be known when *man* is determined.

Let the circle be turned round until the reflexion of the light coming through a narrow slit *A*, in a piece of card-board parallel to the axis of the circle, at the face *ab* of the crystal, is seen in the direction of another parallel line at *B*, and let the vernier be read off; then the circle being turned until the reflexion of the light from *A* by the face *ac* is seen in the same direction *aB*, and the vernier read off again; the difference of the two readings gives the angle through which the circle has been turned, which is the angle *man* between the normals.

The arms by which the crystal is placed in its required position are omitted in the figure, as the section of the crystal, for the sake of distinctness, is drawn disproportionately large compared with the circle. It will be seen that this instrument suffices to measure the angles between the plane and polished surfaces of crystals, however small they may be.

Frauenhofer's Table of Refractive Indices for the seven principal fixed lines of the Spectrum in some kinds of Glass and Liquids.

REFRACTING MEDIUM.	SPECIFIC GRAVITY.	VALUES OF						
		μ(B)	μ(C)	μ(D)	μ(E)	μ(F)	μ(G)	μ(H)
Flint Glass, No. 13 . .	3·723	1·627749	1·629681	1·635036	1·642024	1·648260	1·660285	1·671062
Crown Glass, No. 9 . .	2·535	1·525832	1·526849	1·529587	1·533005	1·536052	1·541657	1·546566
Water	1·000	1·330935	1·331712	1·333577	1·335851	1·337818	1·341293	1·344177
Water, another experiment.	1·000	1·330977	1·331709	1·333577	1·335849	1·337788	1·341261	1·344162
Solution of Potash . .	1·416	1·399629	1·400515	1·402805	1·405632	1·408082	1·412579	1·416368
Oil of Turpentine . .	0·885	1·470496	1·471530	1·474434	1·478353	1·481736	1·488198	1·493874
Flint Glass, No. 3 . .	3·512	1·602042	1·603800	1·608494	1·614532	1·620042	1·630772	1·640373
Flint Glass, No. 30 . .	3·695	1·623570	1·625477	1·630585	1·637356	1·643466	1·655406	1·666072
Crown Glass, No. 13 . .	2·535	1·524312	1·525299	1·527982	1·531372	1·534337	1·539908	1·544684
Crown Glass, letter M .	2·756	1·554774	1·555933	1·559075	1·563150	1·566741	1·573535	1·579470
Flint Glass, No. 23 Prism of 60° 15′ 42″ .	3·724	1·626596	1·628469	1·633667	1·640495	1·646756	1·658848	1·669686
Flint Glass, No. 23 Prism of 45° 23′ 14″ .	3·724	1·626564	1·628451	1·633666	1·640544	1·646780	1·658849	1·669680

A Table of the Refractive Indices of some substances for the mean rays, selected from the large Table in Sir John Herschel's "Treatise on Light" in the 'Encyclopædia Metropolitana.'

SUBSTANCES.	VALUE OF μ.	SUBSTANCES.	VALUE OF μ.
Vacuum	1·	Beryl	1·598
Hydrogen Gas	1·000138	Topaz	1·610
Oxygen Gas	1·000272	to	1·652
Atmospheric Air	1·000294	Oil of Cassia	1·641
Nitrogen Gas	1·000300	Tourmaline	1·668
——		Chrysoberyl	1·760
Tabasheer	1·1111	Sapphire	1·768
to	1·1825	to	1·794
Water	1·336	Ruby	1·779
Alcohol	1·372	Arsenic	1·811
Oil of Olives	1·470	Sulphate of Lead	1·925
Gum Arabic	1·512	Zircon least	1·961
Camphor	1·500	Zircon greatest	2·015
Plate Glass	1·500	Sulphur	1·958
to	1·550	to	2·148
Crown Glass	1·525	Glass of Antimony	1·980
to	1·544	to	2·216
Flint Glass	1·576	Phosphorus	2·125
to	1·642	to	2·260
Canada Balsam	1·528	Diamond	2·439
to	1·549	to	2·755
Emerald	1·585	Chromate of Lead least	2·500
		Chromate of Lead greatest	2·974
		Realgar	2·549

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